A NEW DIMENSION-REDUCTION METHOD FOR COMPLEX DYNAMICAL NETWORKS

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SIAM Workshop on Network Science 2018
July 12-13 · Portland

Summary

We introduce a new dimension-reduction method to describe the large scale behaviour of dynamical processes running on networks, primarily based on the spectral properties of the weighted adjacency matrices that characterize the interactions on the networks. The structural complexity of the networks is used to naturally set the adequate dimensionality of the reduced system. We present and compare three variants of our method. We show that our approximation scheme, even when forced to produce one-dimensional reduced systems, always gives a better description of the dynamics than the one proposed by Gao *et al.*[1].

Introduction

Dynamics of large complex networks can sometimes be modeled as simpler and lower dimensional systems. These reduced systems, if properly inferred, should provide intuitive insights about the global behaviour of the systems and help predict their dynamical resilience or breakdown.

We consider a network of N nodes encoded by the weighted and directed adjacency matrix $\boldsymbol{W} = (w_{ij})$, where the element $w_{ij} \geq 0$ indicates the strength of the directed interaction from node j to node i. Node i has an activity $x_i \in \mathbb{R}$ that evolves according to

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{N} w_{ij} G(x_i, x_j). \tag{1}$$

Recent studies suggest that the global equilibrium states of such a N-dimensional system can be reduced to a one-dimensional universal function [1]. Two effective structural and activity parameters can then be extracted to describe the evolution of the system. While this is a good approximation for uncorrelated network structures [2], it fails when degree correlations become important.

Generalizing the dimension-reduction

To reduce the dimensionality of the dynamical system (1), we introduce new weighted averages that describe the global dynamics propagating on the network and the large-scale structure of the network. For each k, let $\boldsymbol{a}^{(k)} = (a_1^{(k)}, \ldots, a_N^{(k)})$ be a discrete probability distribution, i.e., $a_i^{(k)} \geq 0 \; \forall \, i$ and $\sum_i a_i^{(k)} = 1$. These probability distributions allow to define weighted average activities, $\langle x \rangle^{(k)} = \sum_{j=1}^{N} a_j^{(k)} x_j$, as well as the weighted average in-degrees, $\langle w \rangle^{(k)} = \sum_{j=1}^{N} a_j^{(k)} w_j^{\text{in}}$.

We can show that the dynamics of the weighted average activities $\langle x \rangle^{(k)}$, describing the system (1), is approximately governed by the following system:

$$\begin{aligned}
\dot{\langle x \rangle}^{(0)} &= F(\langle x \rangle^{(0)}) + \langle w \rangle^{(0)} G(\langle x \rangle^{(1)}, \langle x \rangle^{(1)}) \\
\dot{\langle x \rangle}^{(1)} &= F(\langle x \rangle^{(1)}) + \langle w \rangle^{(1)} G(\langle x \rangle^{(2)}, \langle x \rangle^{(2)}) \\
&\vdots
\end{aligned}$$

where the distributions are transformed according to

$$\langle w \rangle^{(k)} \boldsymbol{a}^{(k+1)} = \boldsymbol{W}^T \boldsymbol{a}^{(k)}. \tag{2}$$

Transformation (2) is a well-defined map on the space of probability distributions since it preserves both normalization (i.e., $\sum_i a_i^{(k+1)} = 1$ if $\sum_i a_i^{(k)} = 1$) and positiveness (i.e., $a_i^{(k+1)} \geq 0$ if $a_i^{(k)} \geq 0$).

At first glance, the new dynamical system is unsolvable; it contains an infinite number of dynamical variables. However, an appropriate choice of the initial probability distribution $\boldsymbol{a}^{(0)}$ enables us to close the set of differential equations and get a d-dimensional system. We have developed three procedures to choose $\boldsymbol{a}^{(0)}$, all of them leading to a d–dimensional reduced system: (i) the d–period method (ii) the power iteration method and (iii) the eigenvector composition method.

The d–period method consists in choosing $\boldsymbol{a}^{(0)}$ such that $\boldsymbol{a}^{(d+1)} = \boldsymbol{a}^{(0)}$. In doing so, the d-th differential equation closes the system since $\dot{\langle x \rangle}^{(d)} = f(\langle x \rangle^{(d)}, \langle x \rangle^{(0)})$. We achieve this by letting $\boldsymbol{a}^{(0)}$ be the positive eigenvector of the d-th power of $\boldsymbol{W}^T$.

We can show that the d–period method always works for strongly connected networks, i.e., networks in which there is a path between each pair of nodes. The case $d = 1$ is of particular interest since it always leads to a one-dimensional reduced system in which the distribution $\boldsymbol{a}^{(0)}$

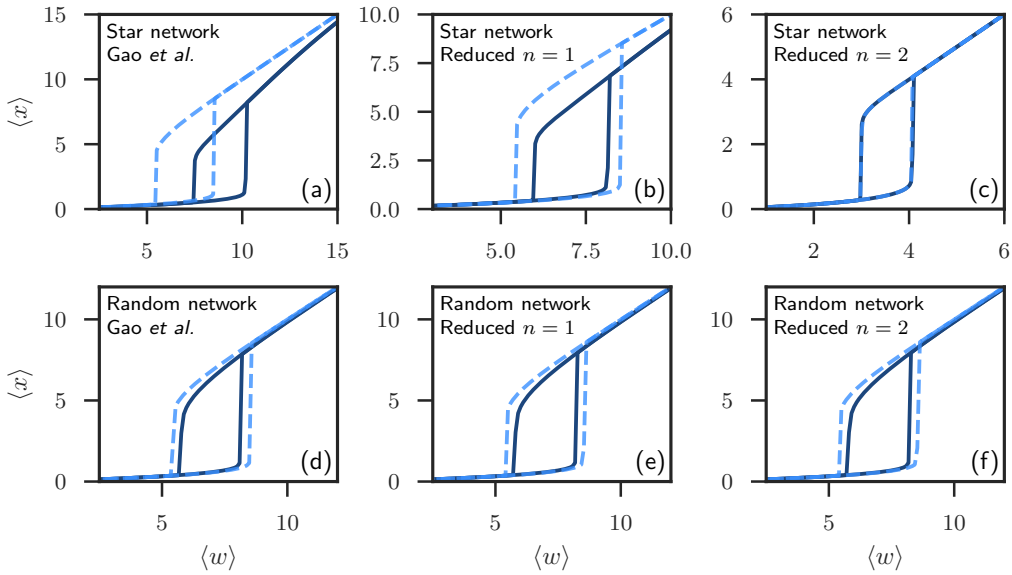

Figure 1: Weighted average activity at equilibrium $\langle x \rangle$ as a function of the weighted average input degree $\langle w \rangle$ using Gao et al. reduction (first column) and our approach with $n = 1$ and $n = 2$ global variables (second and third columns) for star networks of 6 nodes and Gilbert random graph of 100 nodes of density $p = 0.1$. Dashed lines are the predictions of each approach while full lines are obtained from numerical simulations on networks with neural dynamics.

is identical to the eigenvector centrality. The formalism proposed by Gao *et al.* is an approximation of the $(d = 1)$–period method.

In the power iteration method, we set $\boldsymbol{a}^{(0)}$ as the uniform probability distribution, i.e., $a_i^{(0)} = 1/N$ for all i. Then, as we apply the transformation (2), $\boldsymbol{a}^{(k)}$ aligns with the dominant eigenvector of $\boldsymbol{W}^T$. When the convergence $\|\boldsymbol{a}^{(k+1)} - \boldsymbol{a}^{(k)}\| < \epsilon$ is reached, the k-th equation of the reduced system can safely be approximated by $\langle \dot{x} \rangle^{(k)} = f(\langle x \rangle^{(k)})$. In practice, this method should be used only to obtain the evolution of the uniform activity average.

The eigenvector composition method consists in choosing $\boldsymbol{a}^{(0)}$ as a linear composition of d dominant and linearly independent eigenvectors of $\boldsymbol{W}^T$. In doing so, $\boldsymbol{a}^{(d)} = \sum_{j=0}^{d-1} c_j \boldsymbol{a}^{(j)}$ is simply a linear composition of the constructed probability distribution.

Results

We apply our formalism to different network structures (Fig. 1) on which the activity evolves according to a well-known dynamics in computational neuroscience [3],

$$\dot{x}_i = -x_i + \sum_{j=1}^{N} w_{ij} \sigma(x_j - \mu)$$

where μ is a parameter and $\sigma(\cdot)$ is the sigmoid function. For random networks, the 1–period method performs as well as the Gao *et al.* formalism. As expected by the spectral analysis of the weighted adjacency matrix, higher dimensions of the reduction do not improve the description.

An impressive demonstration of the power of our formalism is given by star graphs where a single core node is connected to $N-1$ periphery nodes. The one-dimensional system of Gao *et al.* overestimates the activation of the core node so that large discrepancies are visible, even for small graphs $N = 6$, and increase with the size of the graph. Using our formalism, we describe *exactly* the star graph using a two-dimensional system. Moreover, we show that $\langle x \rangle^{(0)}$ and $\langle x \rangle^{(1)}$ describe the activity of the core node and the periphery nodes respectively.

References

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