Synchronization dynamics on the stochastic block model

V. Thibeault[1,2], X. Roy-Pomerleau[1,2], G. St-Onge[1,2], J-G. Young[1,2], L.J. Dubé[1,2] and P. Desrosiers[1,2,3]

1. Département de physique, de génie physique et d'optique, Université Laval, Québec (QC), G1V 0A6, Canada
2. Centre interdisciplinaire de modélisation mathématique de l'Université Laval, Québec (QC), G1V 0A6, Canada
3. Centre de recherche CERVO, Québec (QC), G1J 2G3, Canada

Problem and contributions

Synchronization has attracted much attention over the last decade, partly due to the intriguing chimera states, a state where coherence and incoherence cohabit. Yet the effects of **network structure** on the various **synchronization states** are still mysterious. To decipher the impact of mesoscale structure of networks on synchronization, we study, both theoretically and numerically, the dynamics of Kuramoto oscillators interacting on networks drawn from the **stochastic block model (SBM)**. This allows to:

- Find new regions in the structural parameter space where chimeras exist;
- Predict the critical coupling at which synchronization can occur;
- Measure the effects of structure on the chaotic behavior of chimeras.

Kuramoto model

There are n oscillators interacting on a network of adjacency matrix A and evolving according to

$$\dot{\theta}_i = \omega_i + \frac{\sigma}{n}\sum_{i=1}^{n} A_{ij}\sin(\theta_j - \theta_i - \alpha)$$

- θ_i is the phase of the ith oscillator;
- ω_i is the natural frequency of the ith oscillator;
- σ is the coupling constant;
- α is a phase-lag.

Structure-function parameters

Structural parameters

- Two **asymmetric** blocks (communities) of sizes

$$n_1 = f_1 n \quad\text{and}\quad n_2 = f_2 n.$$

- Block indices

$$B_1 = \{1, ..., n_1\} \quad\text{and}\quad B_2 = \{1, ..., n_2\}.$$

- Asymmetry is characterized by the parameters

$$f = \frac{f_1}{f_2} = \frac{n_1}{n_2} \quad\text{and}\quad \beta = \frac{n_1(n_1-1)+n_2(n_2-1)}{n(n-1)}.$$

- Probability of connecting nodes inside and outside the same block: p_{in} and p_{out}, respectively.
- **Density space** has coordinates

$$\rho = \beta p_{in} + (1-\beta)p_{out}, \qquad \Delta = p_{in} - p_{out}.$$

Synchronization measures

- **Macroscale order parameter**

$$R(t) = \left|\frac{1}{n}\sum_{j=1}^{n} e^{i\theta_j(t)}\right|$$

It describes the global behavior of the oscillators.

- **Mesoscale order parameter**

$$R_\mu(t) = \left|\frac{1}{n_\mu}\sum_{j\in B_\mu} e^{i\theta_j(t)}\right|$$

It describes the behavior of the oscillators in each block.

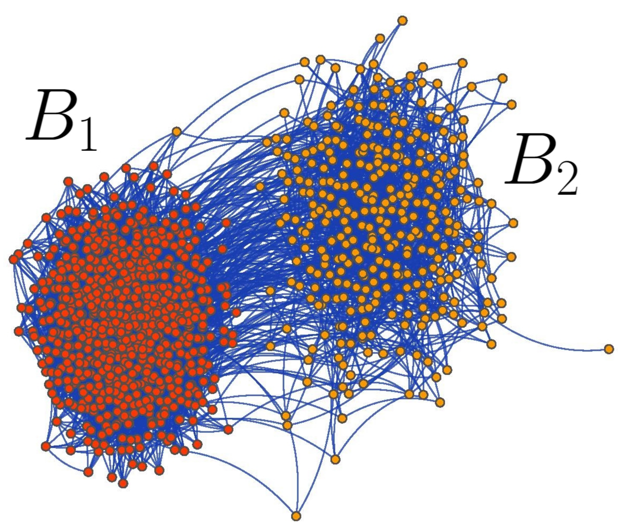

Fig. 1: Typical network in the two-block SBM
There are $n = 500$ nodes with $n_1 = 300$ and $n_2 = 200$ ($f = 1.5$, $\beta \approx 0.52$). These parameters were also used in Fig. 2, 3 & 4.

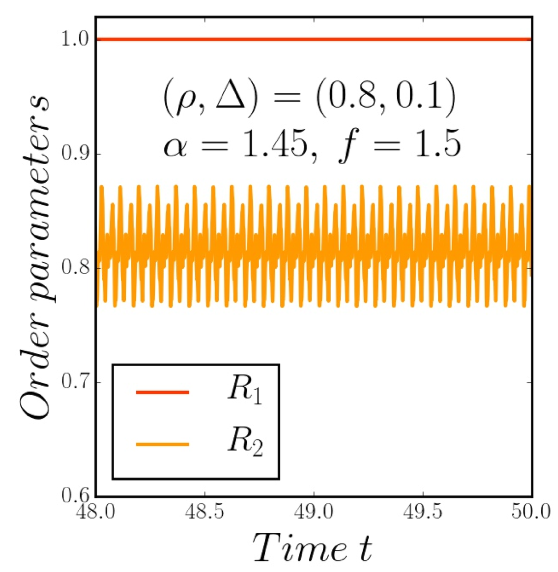

Fig. 2: What is a chimera state?
It is a state where one block is synchronized ($R_1 = 1$) and the other block is partially synchronized ($R_2 \equiv R_I < 1$).

- Is it possible to have chimeras for a particular graph drawn from the SBM ensemble?
  Yes, if we relax the definition. We observe states where neither $R_1(t)$ and $R_2(t)$ are perfectly synchronized, but where these order parameters oscillate around different values. Note also that in the dense regime $A \approx \langle A\rangle$, the mean SBM adjacency matrix. For this reason, simulations in Fig. 2, 3 & 5 use $\langle A\rangle$ instead of A.
- How to choose the initial conditions to get a chimera?
  We need to draw multiple initial conditions from a uniform distribution for each point in the parameter space. The number of observed chimeras per initial condition is not equal everywhere.
- How does the size of the chimera region (see Fig. 3) evolve as the structural parameters change?
  It evolves in a nonlinear way when we increase f. For $\alpha = 1.45$, the area of the chimera region in the density space is smaller for $f = 1$ than for $f = 1.5$. However, the area for $f = 1.5$ is bigger than for $f = 2$.

New chimera regions

To obtain the analytical results we

1. Take the continuum limit;
2. Introduce a continuity equation;
3. Apply the Ott-Antonsen Ansatz [1];
4. Perform a linear stability analysis.

The ensuing **reduced dynamics** is then characterized by two time-dependent variables: $r_I(t)$ as the mesoscale order parameter of the incoherent community and $\phi(t) = \phi_1 - \phi_2$ as the mesoscale phase difference. A linear relationship emerges between Δ and ρ

$$\Delta = \frac{\rho\,[r_I\cos\alpha + f\cos(\phi-\alpha)]}{\beta f\cos(\phi-\alpha) - (1-\beta)r_I\cos\alpha},$$

which we then use to detect possible bifurcations (here Hopf and saddle-node). The results are validated numerically in Fig. 3.

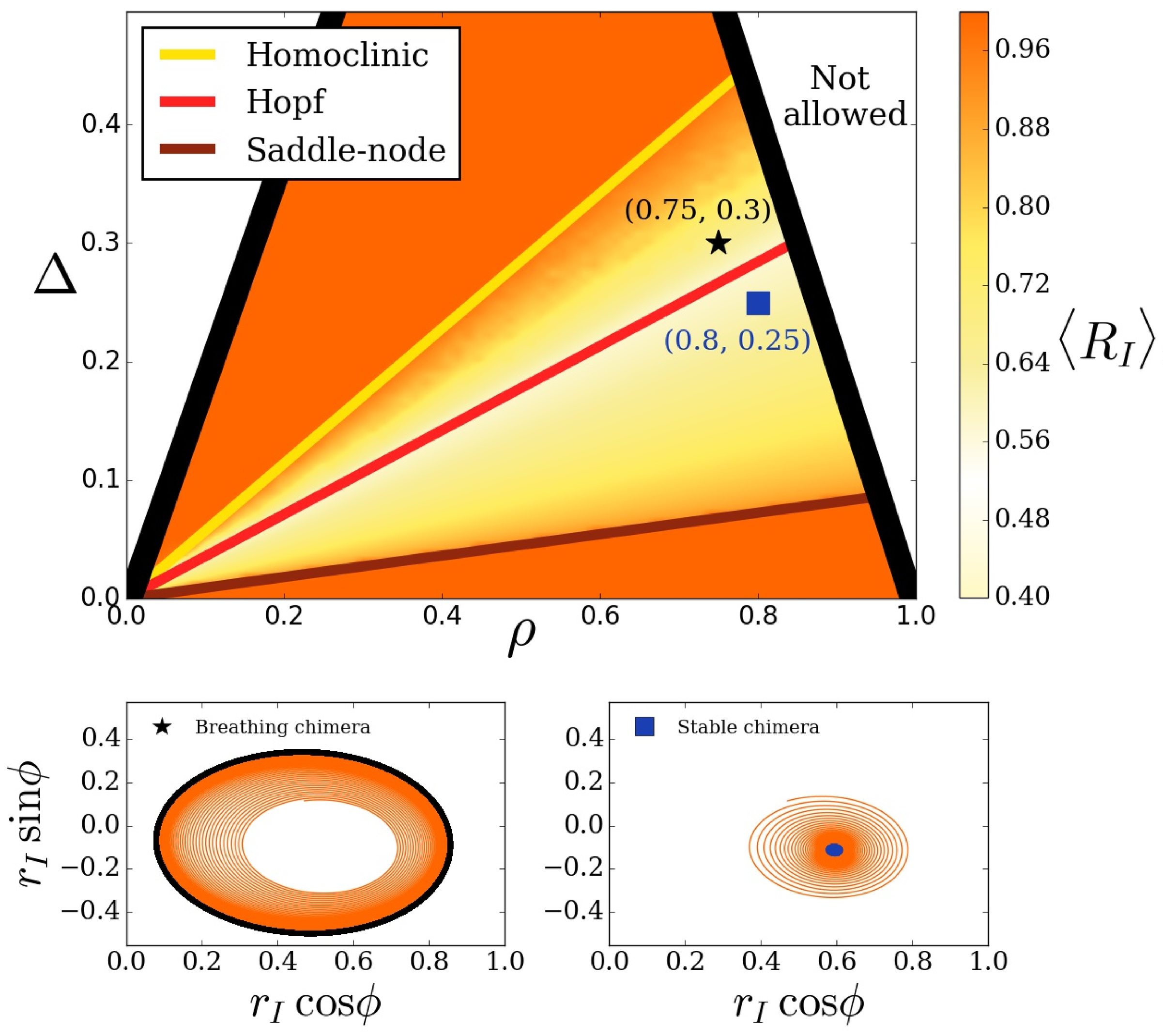

Fig. 3: Chimeras in the density space
(Top) Each point represents the value of $\langle R_I\rangle_t$, the time average of the mesoscale order parameter of the incoherent community. Parameters: $f = 1.5$, $\alpha = 1.45$, $\omega_i = 0\ \forall i$. (Bottom left) Breathing chimera at $(\rho, \Delta) = (0.75, 0.3)$ obtained from the reduced dynamics. (Bottom right) Stable chimera at $(\rho, \Delta) = (0.8, 0.25)$ obtained from the reduced dynamics. Bottom figures trajectories have the same initial condition ($r_I(0)$, $\phi(0)$) = (0.48, 0.24).

What about bifurcations ?

- **Homoclinic**: A breathing chimera (limit cycle) encounters a saddle point. It is obtained numerically from the integration of the Kuramoto model.
- **Hopf (supercritical)**: A stable chimera (fixed point) loses stability and trajectories converge to a stable breathing chimera. It is obtained analytically by taking the trace of the Jacobian matrix of the reduced dynamics equal to 0.
- **Saddle-node**: A stable chimera is created or destroyed. It is obtained analytically by taking the determinant of the Jacobian matrix of the reduced dynamics equal to 0.

Critical coupling and $\langle A\rangle$ spectrum

When the coupling σ is below a critical value σ_{crit}, the oscillators cannot synchronize. To predict σ_{crit}, we use the exact expression for the **largest eigenvalue** [2] of the mean SBM adjacency matrix. The theoretical value depends mainly on the structural parameter:

$$\sigma_{crit} = \frac{4\pi^{-1}g(0)^{-1}}{p_{in} + \sqrt{(f_1-f_2)^2 p_{in}^2 + 4f_1 f_2 p_{out}^2}}$$

where $g(\omega)$ is the density of natural frequencies (ex. Lorentzian, Gaussian). This prediction is validated numerically in Fig. 4.

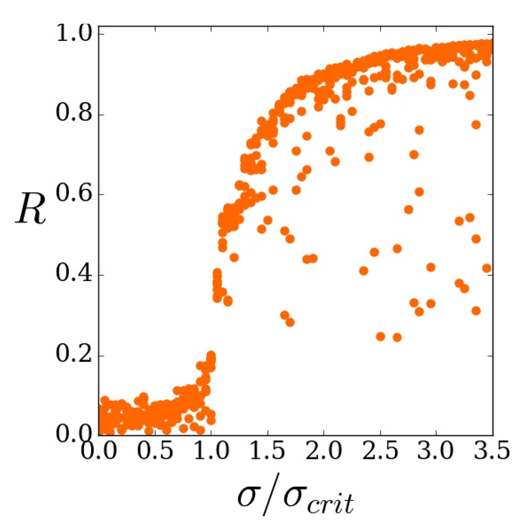

Fig. 4: Transition to global synchronization
Time-averaged global order parameter R versus the coupling ratio σ/σ_{crit}. Each point represents a different simulation with randomly chosen adjacency matrices and initial conditions.

Chaos in chimera dynamics

The chimera states obtained from the same initial conditions only exist in a very restricted region of the density space. To gain more insight on the evolution of the chimeras in that region, we investigate the variation, along the ρ axis, of the **extrema of the mesoscale order parameter** [3] as well as the maximum Lyapunov exponent. As shown in Fig. 5, the extrema exhibit different behaviors:

- Stable periodic orbits;
- Period doubling;
- Chaos.

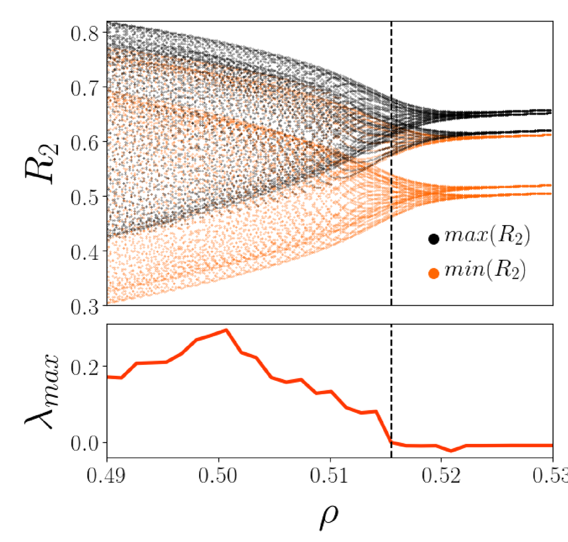

Fig. 5: Chaos in the extrema dynamics
(Top) Extrema of the mesoscale dynamics versus ρ. (Bottom) Largest Lyapunov exponent in the evolution of the maxima of $R_2 = R_I$ for each density ρ. The vertical dotted line is at $\rho = 0.5155$ and represents the transition from chaotic to periodic solutions. Parameters: $n_1 = n_2 = 128$, $f = 1$, $\beta \approx 0.5$, $\alpha = 1.47$, $\omega_i = 0\ \forall i$ and $\Delta = 0.28$.

Calcul Québec · CRSNG NSERC · Sentinelle Nord · Fonds de recherche Nature et technologies Québec

[1] T. Kotwal, X. Jiang, and D. Abrams. *Phys. Rev. Lett.*, **119**(26), p.264101 (2017).
[2] F. Rodrigues, T. Peron, P. Ji, and J. Kurths. *Physics Reports*, **610**, p.1-98 (2016).
[3] C. Bick, M. Panaggio, and E. Martens. arXiv: 1802.05481v1 (2018).