Dimension reduction of high-dimensional dynamics on networks with adaptation

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NERCCS 2021

Dynamica

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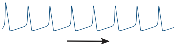
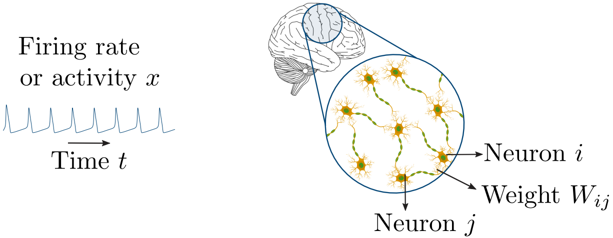
Emergence of collective phenomena (synchronization)

https://www.youtube.com/watch?v=tRPuVAVXk2M


Firing rate
or activity x
Time t


Firing rate
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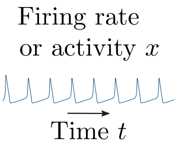
Time t
Neuron i
Weight W_{ij}
Neuron j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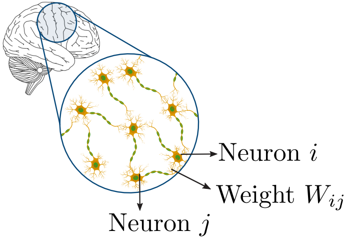
Neuron i
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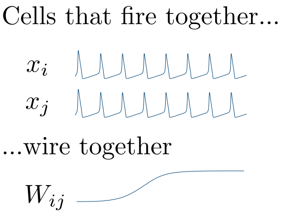
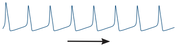
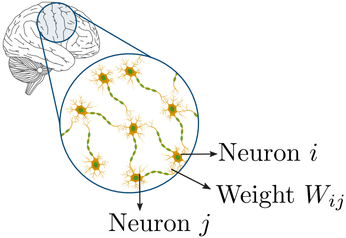
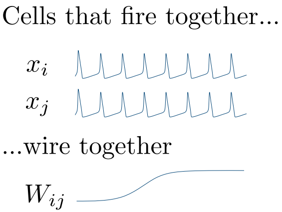

Cells that fire together...
x_i
x_j
...wire together
W_{ij}

Firing rate
or activity x







Nonlinear activity dynamics + **Complex network** + **Nonlinear adaptation (plasticity)**

$$\frac{dx_i}{dt} = F(x_i) + G(x_i, \textstyle\sum_{j=1}^{N} W_{ij} x_j)$$

$$\frac{dW_{ij}}{dt} = H(x_i, x_j, W_{ij})$$

Complete dynamics

$N \gg 1$

$$\frac{dx_i}{dt} = F(x_i) + G(x_i, \textstyle\sum_{j=1}^{N} W_{ij} x_j)$$

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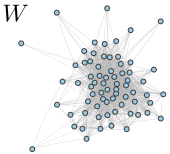
$$i, j \in \{1, ..., N\}$$



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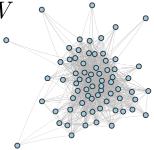
W



Reduced dynamics

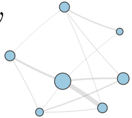
$n \ll N$

$$\frac{d\mathcal{X}_\mu}{dt} = f(\mathcal{X}_1, ..., \mathcal{X}_n, \mathcal{W})$$

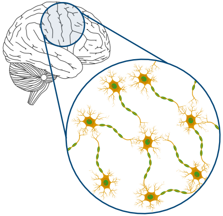
$$\frac{d\mathcal{W}_{\mu\nu}}{dt} = h(\mathcal{X}_\mu, \mathcal{X}_\nu, \mathcal{W})$$

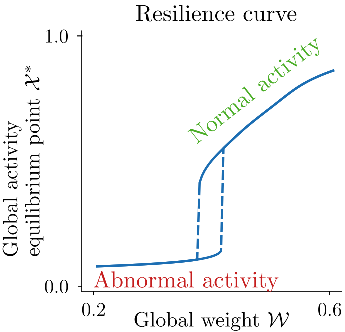
$$\mu, \nu \in \{1, ..., n\}$$

$\mathcal{W}$



Why dimension reduction ?





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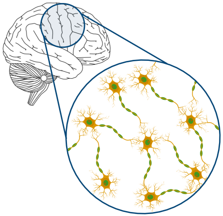

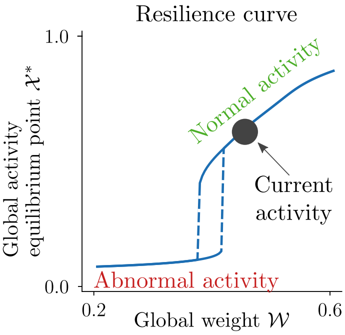

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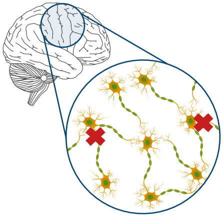
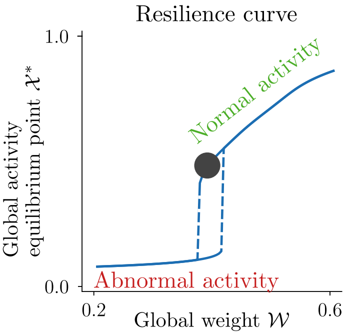
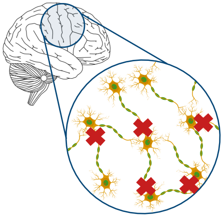
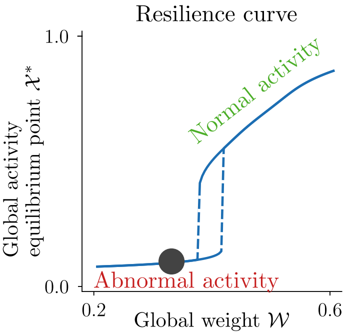
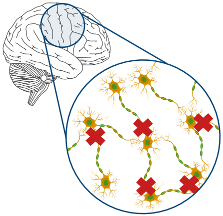


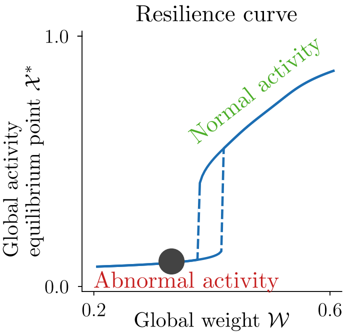





Resilience curve
1.0
0.0
Global activity
equilibrium point $\mathcal{X}^*$
Normal activity
Abnormal activity
0.2
0.6
Global weight $\mathcal{W}$

Why dimension reduction ?





Dimension reduction allows to ...

- find meaningful global variables $\mathcal{X}_\mu, \mathcal{W}_{\mu\nu}$;
- get analytical insights on resilience ;
- reduce computational cost.

Contribution

Complete dynamics

$N \gg 1$

$$\frac{dx_i}{dt} = F(x_i) + G(x_i, \textstyle\sum_{j=1}^{N} W_{ij} x_j)$$

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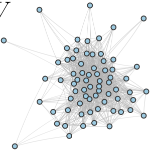
$$i, j \in \{1, \ldots, N\}$$

W



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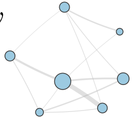
$\mathcal{W}$



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$\mathcal{W}$

We found $n + n^2$ **linear observables (functions, measures,...)**

$$\mathcal{X}_\mu = \sum_{i=1}^{N} M_{\mu i} x_i,$$

$$\mathcal{W}_{\mu\nu} = \sum_{i,j=1}^{N} M_{\mu i} W_{ij} M^\top_{j\nu},$$

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that both depend on only *one* matrix.

M is a $n \times N$ matrix **to be determined**.

Hypothesis

Important neurons contribute strongly to the global activity

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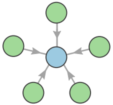
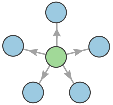
Important neurons contribute strongly to the global activity

Example: Important paper

Important review



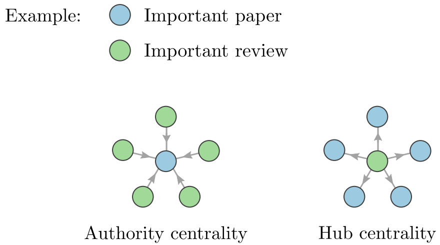
Authority centrality

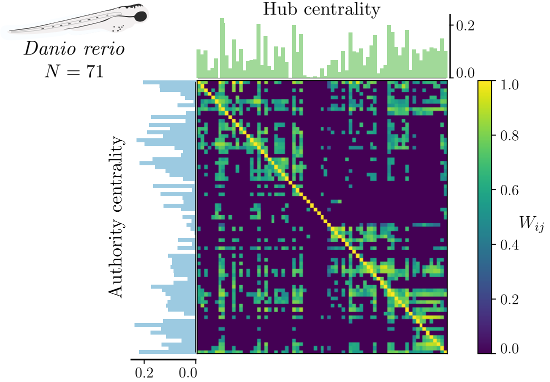


Hub centrality

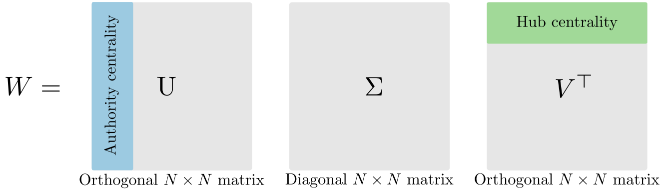
Hypothesis

Important neurons contribute strongly to the global activity





Singular value decomposition (SVD)



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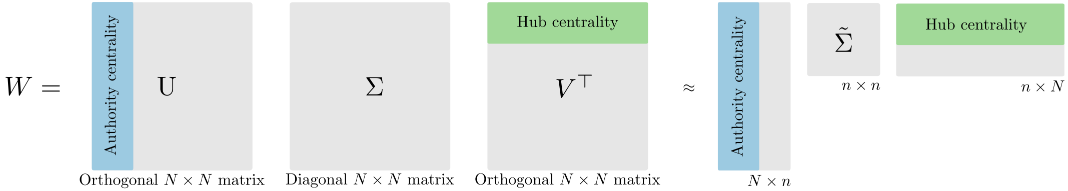

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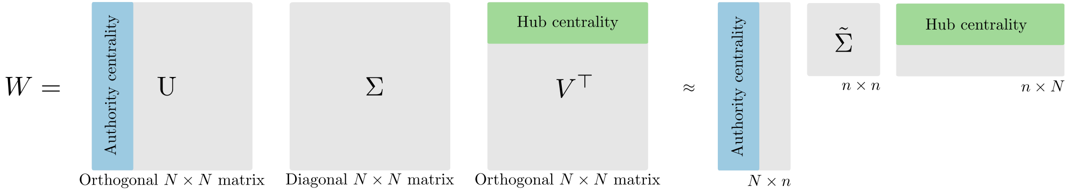
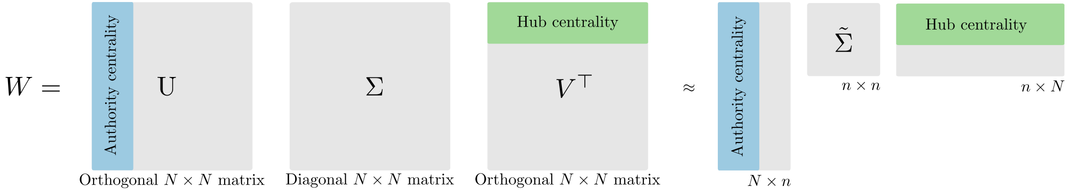


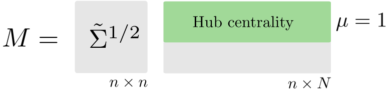
Let $r = \text{rank}(W)$.

If $n \geq r$, the factorization is exact.

If $n < r$, it is the best* approximation of W.

Singular value decomposition (SVD)





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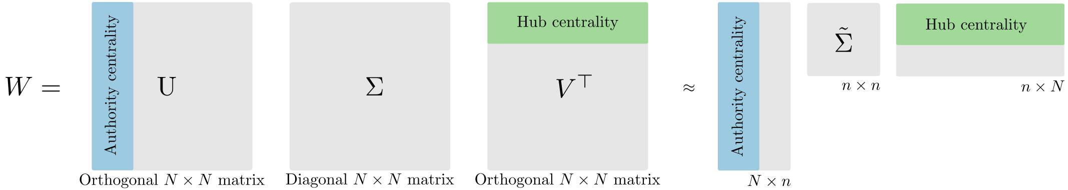

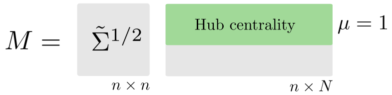

$\Rightarrow$

$$\mathcal{X}_\mu = \sum_{i=1}^{N} M_{\mu i} x_i$$

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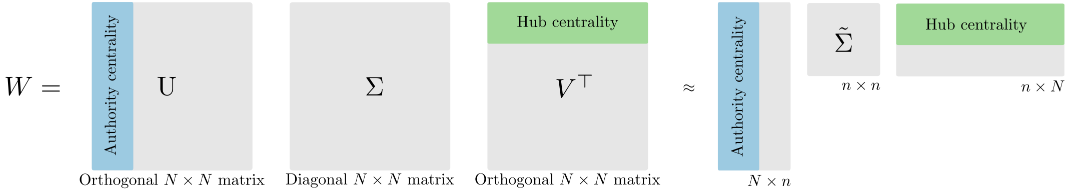
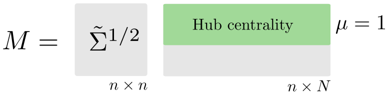
Meaningful at least for $\mu, \nu = 1$!

Singular value decomposition (SVD)





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Meaningful at least for $\mu, \nu = 1$!

$\Rightarrow$ Reduced dynamics

We can combine the observables to get the global activities and weights :

$$\mathcal{X} = a_1\mathcal{X}_1 + ... + a_n\mathcal{X}_n$$

$$\mathcal{W} = b_{11}\mathcal{W}_{11} + b_{12}\mathcal{W}_{12} + ... + b_{nn}\mathcal{W}_{nn}$$

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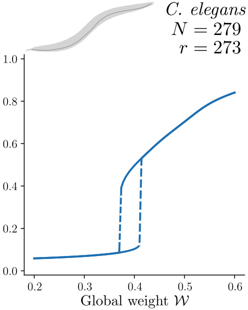
We are ready to get bifurcation diagrams $\mathcal{X}$ vs. $\mathcal{W}$.

Activity dynamics on real networks without plasticity

y-axis

Global activity
equilibrium point $\mathcal{X}^*$

Complete
dynamics



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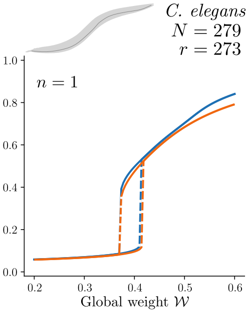

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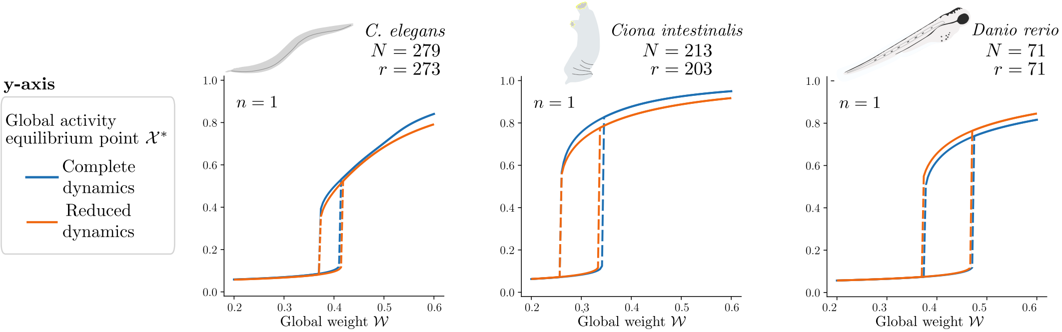

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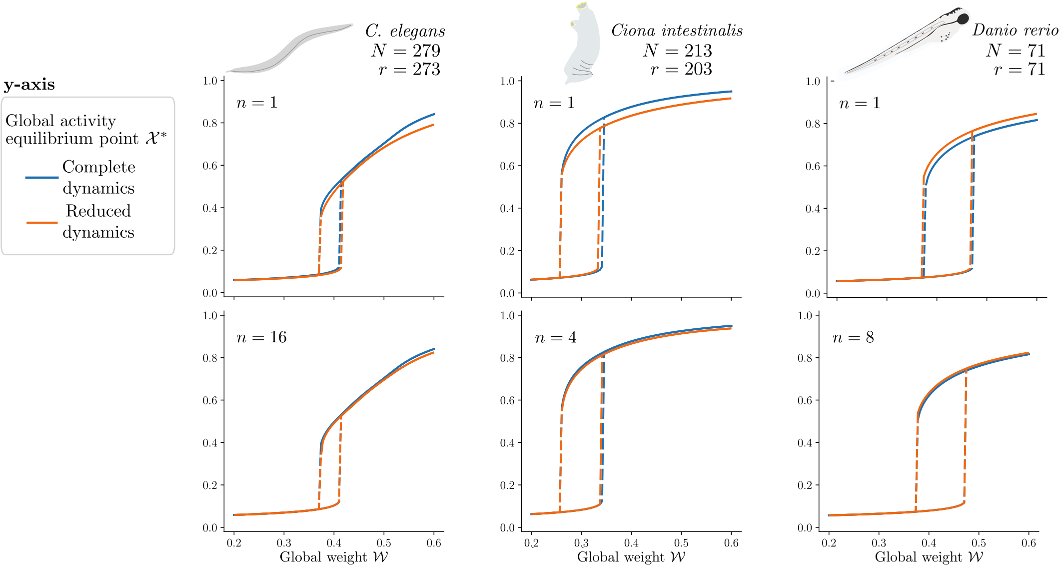
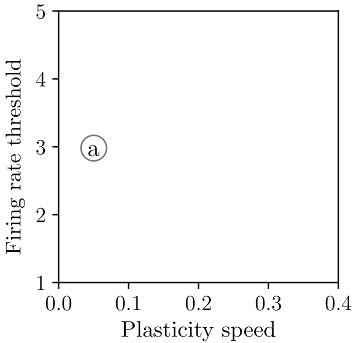
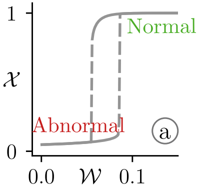


Complete dynamics : 10 200 ODEs

Reduced dynamics : only 3 ODEs





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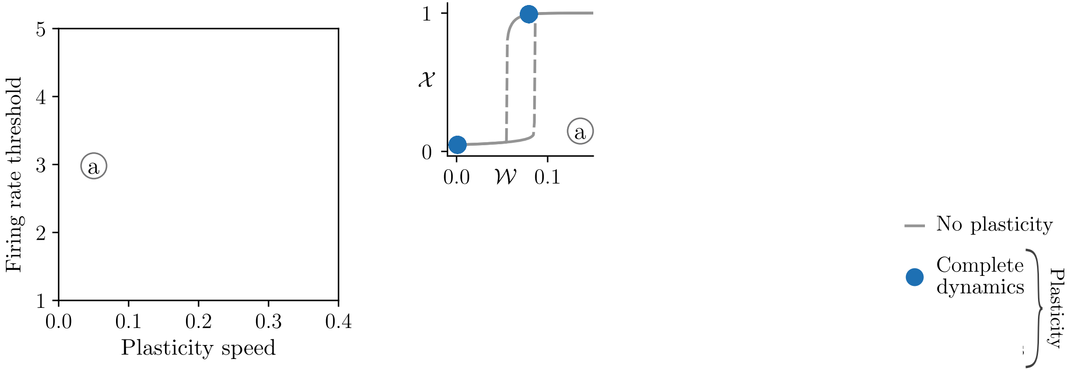

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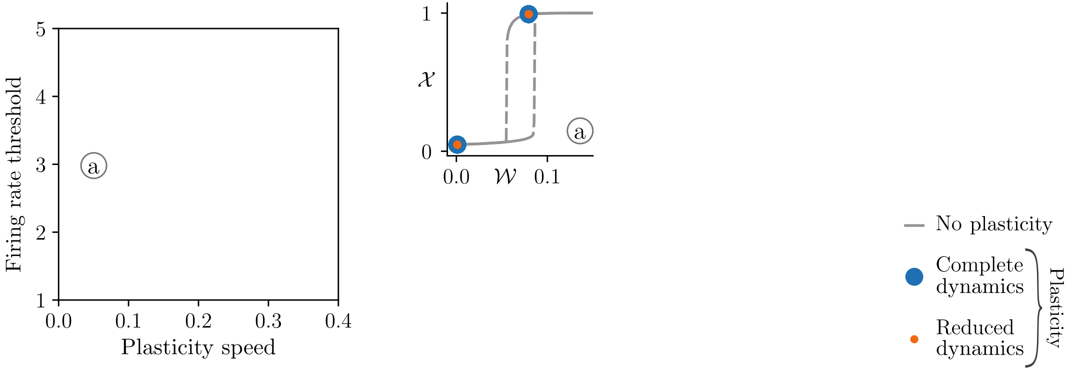
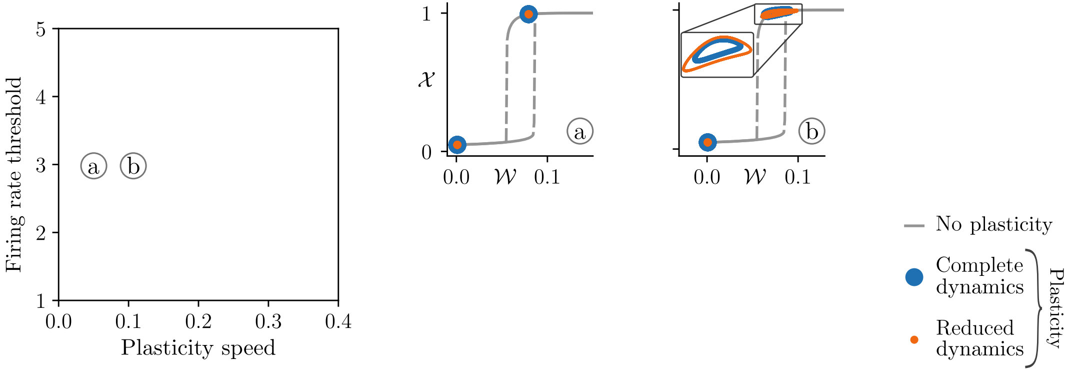


Activity dynamics on an Erdős-Rényi network with plasticity

Complete dynamics : 10 200 ODEs

Reduced dynamics : only 3 ODEs



Complete dynamics : 10 200 ODEs

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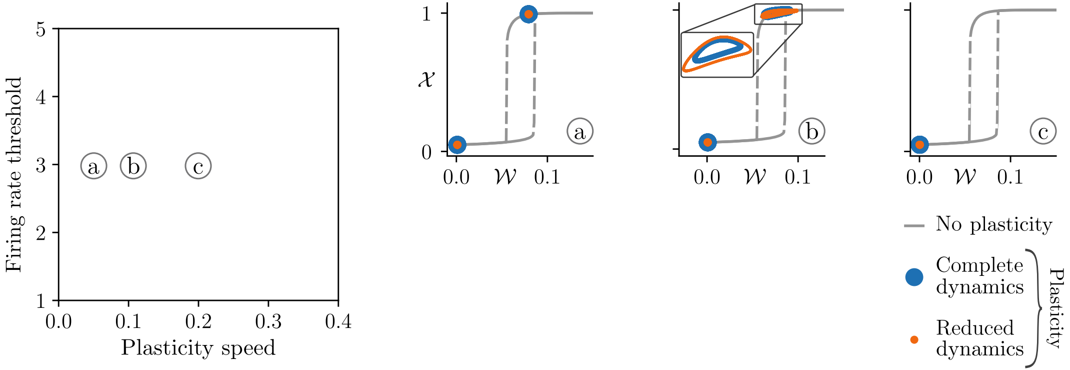
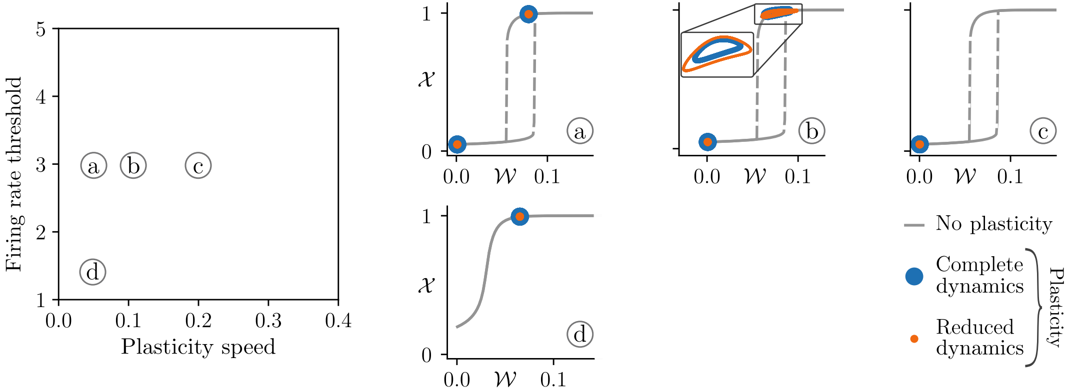


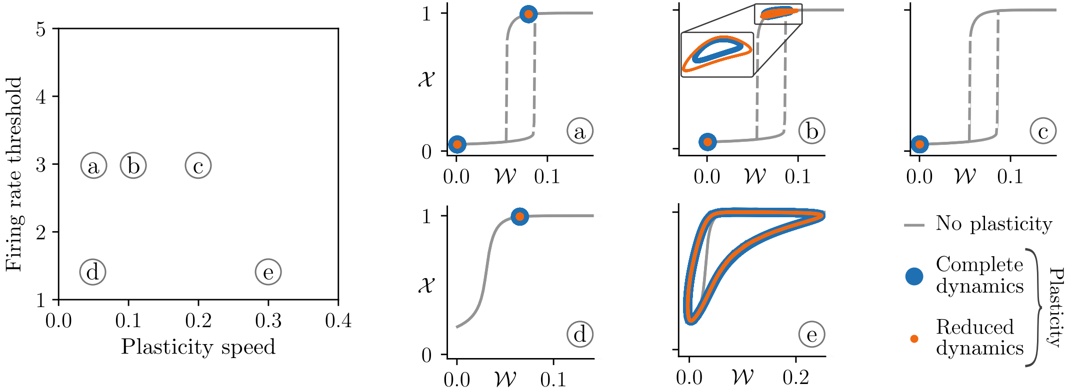
Complete dynamics : 10 200 ODEs

Reduced dynamics : only 3 ODEs



Complete dynamics : 10 200 ODEs

Reduced dynamics : only 3 ODEs



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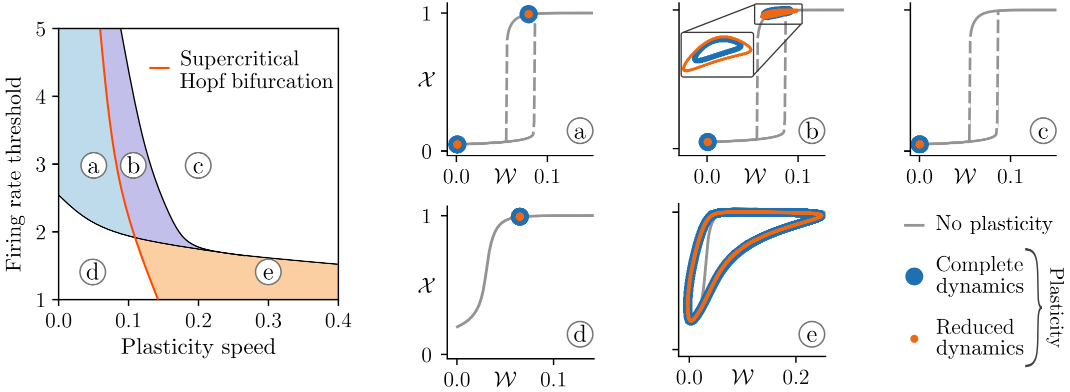

Next steps

- Treat plasticity + real networks;
- Consider inhibitors ($W_{ij} < 0$);
- Get more profound insights on resilience.

Take home messages

- Plasticity leads to *rich* bifurcation diagrams;
- SVD is a powerful and *interpretable* tool for dimension reduction *of dynamics*.

References and acknowledgments

Thank you for your attention !

Thanks to the organizers !

Questions ?

NERCCS 2021

V. Thibeault et al., Phys. Rev. Res. (2020)

E. Laurence et al., Phys. Rev. X (2019)

J. Jiang et al., PNAS (2018)

J. Gao et al., Nature (2016)

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Website : `https://dynamicalab.github.io/`

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Dynamica

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Sentinelle Nord

Calcul Québec

In this model, F is linear and G is a sigmoid function :

$$\tau_x \, \dot{x}_i = -x_i + 1/(1 + e^{-a(y_i - b)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j$$

- x_i : Firing rate of neuron or brain region i
- τ_x : Time scale of the firing rate
- a : Steepness of the activation function
- b : Firing rate threshold

This model is more complex :

$$\tau_x \, \dot{x}_i = -\alpha_i x_i + \beta_i/(1 + e^{-a(y_i - b)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j + \gamma_i$$
$$\tau_w \, \dot{W}_{ij} = D_{ij} x_i x_j (x_i - \theta_i) - \varepsilon W_{ij} \quad \text{with} \quad W_{ij}(0) = d_{ij} D_{ij}$$
$$\tau_\theta \, \dot{\theta}_i = x_i^2 - \theta_i.$$

θ_i : modify the threshold above (below) which the synapse potentiates (depresses).

$\alpha_i, \beta_i, \gamma_i$: distinguish the dynamical behavior of each node i.

$D = (D_{ij})_{i,j=1}^{N}$: structural backbone, $D_{ij} > 0$ if the presynaptic neuron j excites the postsynaptic neuron i, $D_{ij} < 0$ if the presynaptic neuron j inhibits the postsynaptic neuron i, and $D_{ij} = 0$ if no edge exist between neurons i and j.