Predicting synchronization regimes with spectral dimension reduction on graphs

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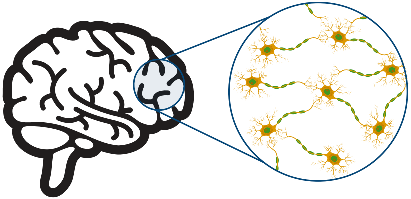
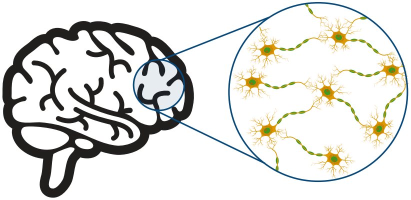


Nonlinear dynamics + **Nonregular graph** $\Longrightarrow$ **Emergent property**

$$\frac{dz_j}{dt} = F(z_j) + \sum_{k=1}^{N} A_{jk} G(z_j, z_k)$$

Complex systems



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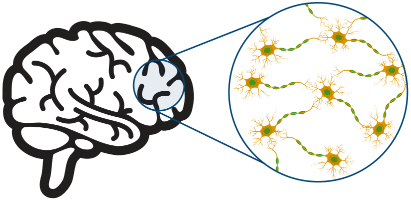

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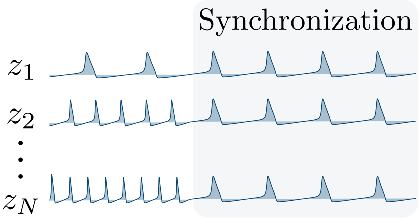

Goal

We want to analyze this type of dynamics

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These dynamical systems are often :

- **very hard** to analyze mathematically
- quite long to integrate numerically

Possible solution : Reduce the number of dimensions of the dynamical system.

Complete dynamics

$N \gg 1$ dimensions

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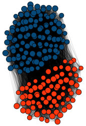
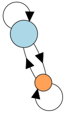


⋮

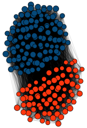
$q \ll N$ dimensions

$$\frac{dZ_\mu}{dt} \approx F(Z_\mu) + \sum_{\nu=1}^{q} \mathcal{A}_{\mu\nu} G(Z_\mu, Z_\nu)$$



Reduced dynamics

Complete dynamics



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Weighted means: $Z_\mu = \sum_{j=1}^{N} M_{\mu j} z_j$

Step 2: **Differentiate with respect to time**

$$\frac{dZ_\mu}{dt} = \sum_{j=1}^{N} M_{\mu j} F(z_j) + \sum_{j,k=1}^{N} M_{\mu j} A_{jk} G(z_j, z_k)$$

Step 3: **Close the system** in terms of the observables

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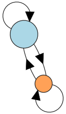

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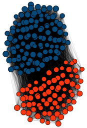

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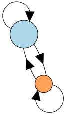

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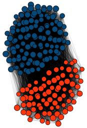

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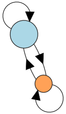

Reduced dynamics

Complete dynamics

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We want to get this weight matrix.

Step 2: **Differentiate with respect to time**

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Reduced dynamics

We don't want to lose the graph properties by doing the dimension reduction.

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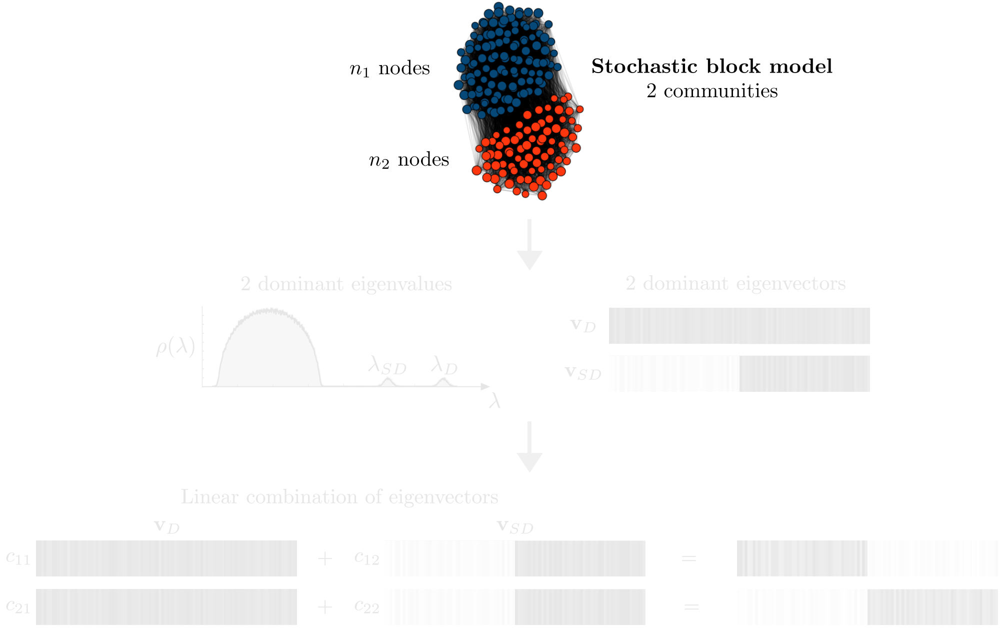
Let's use the spectral graph theory to find M !

See [Laurence et al, *Spectral Dimension Reduction of Complex Dynamical Networks*, Phys. Rev. X (2019)]

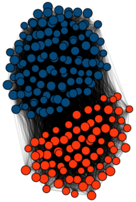
Spectral weight matrix M



Spectral weight matrix M

n_1 nodes

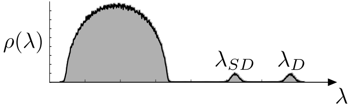
n_2 nodes



Stochastic block model

2 communities

2 dominant eigenvalues



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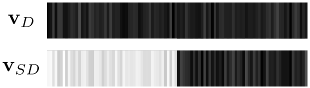
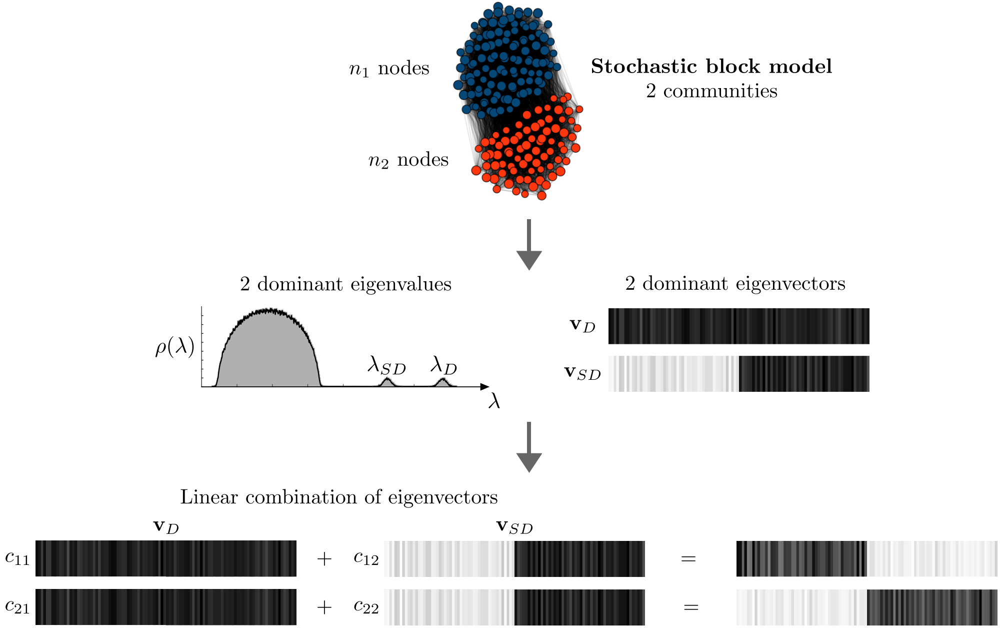


Linear combination of eigenvectors

Spectral weight matrix M



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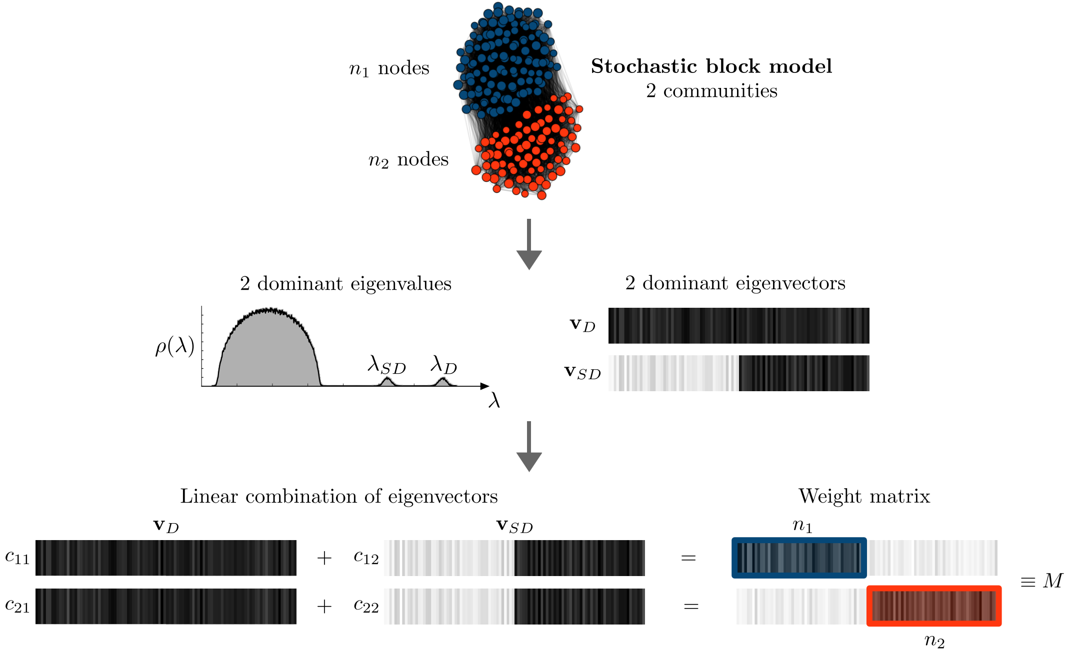
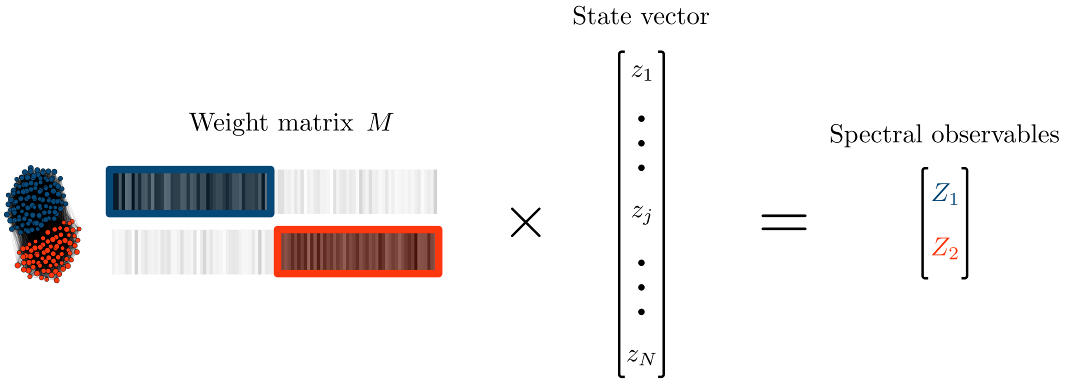
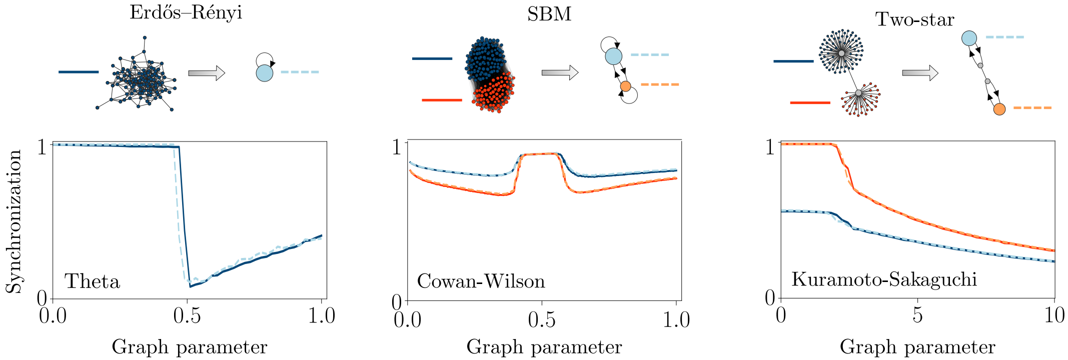



Weight matrix M
State vector
z_1
z_j
z_N
Spectral observables
Z_1
Z_2

Can we predict synchronization regimes with the spectral dimension reduction ?

Synchronization predictions



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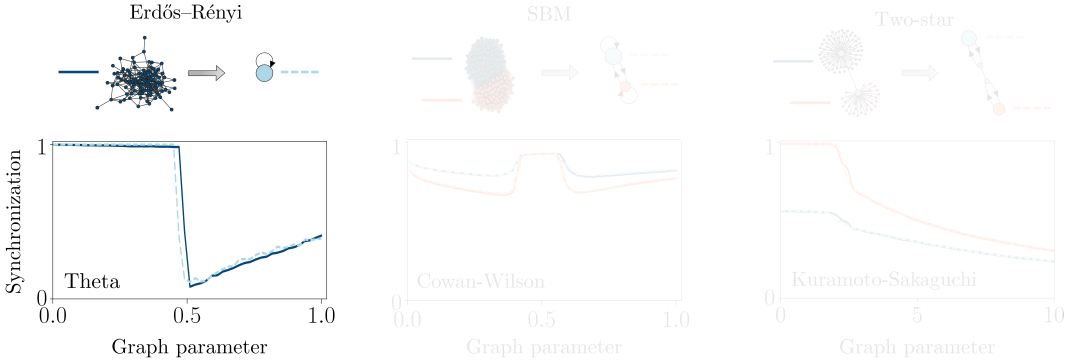

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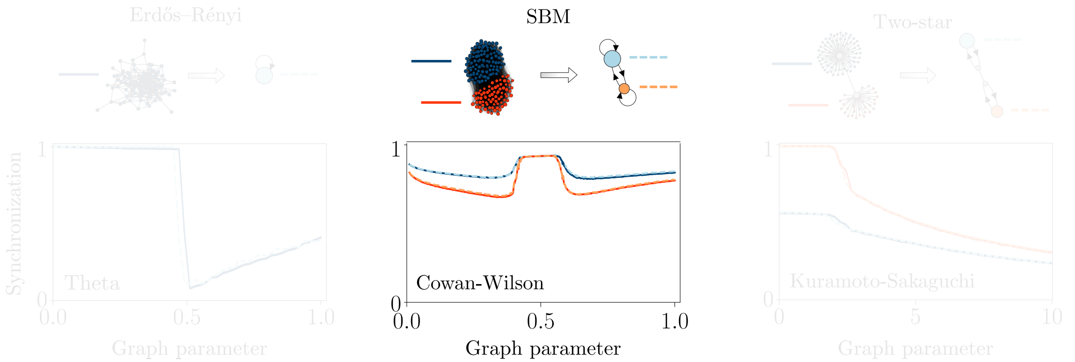

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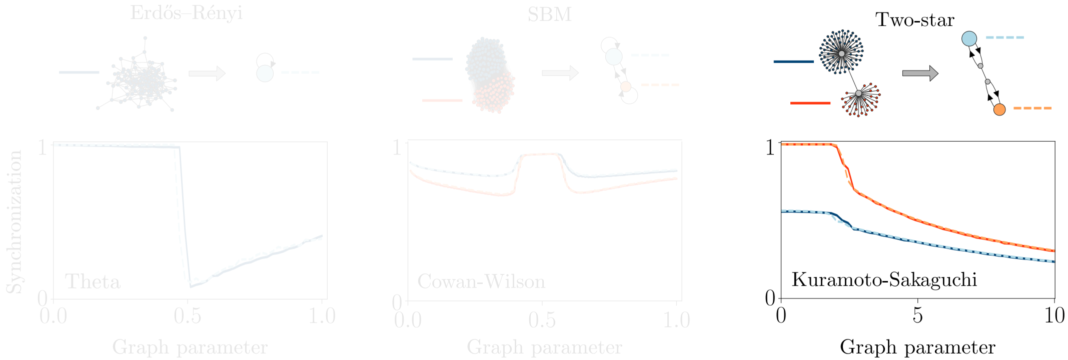
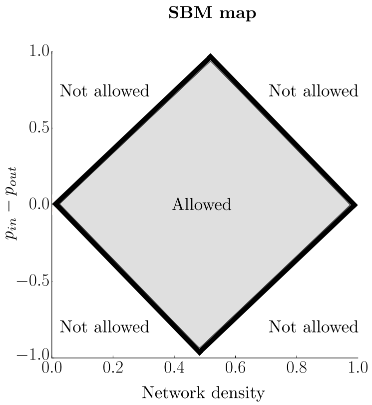
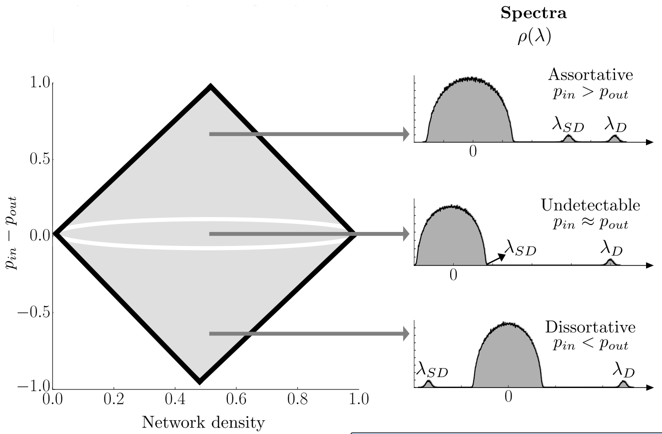


Decipher the influence of the SBM on synchronization

Synchronization in the Cowan-Wilson model

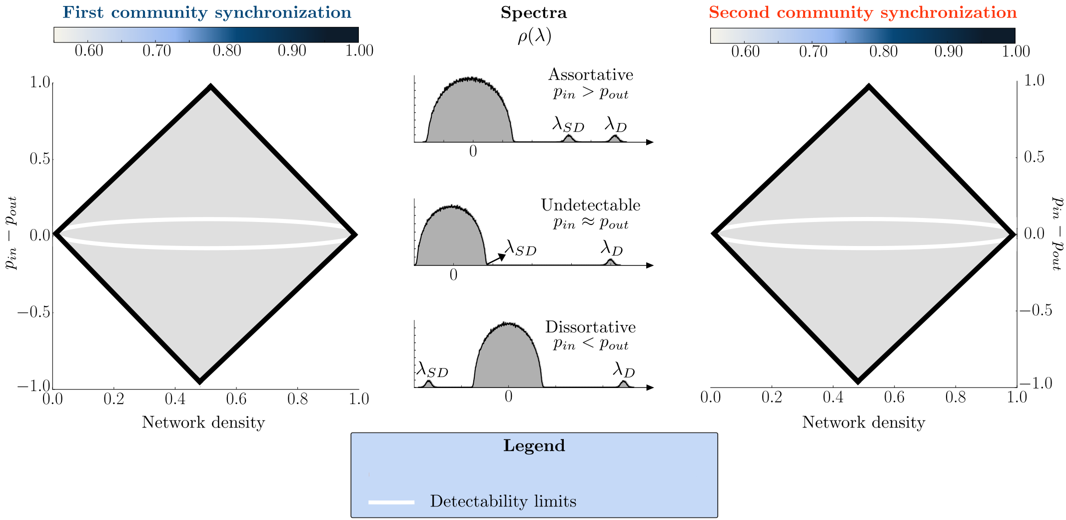




Legend

Detectability limits

Synchronization in the Cowan-Wilson model



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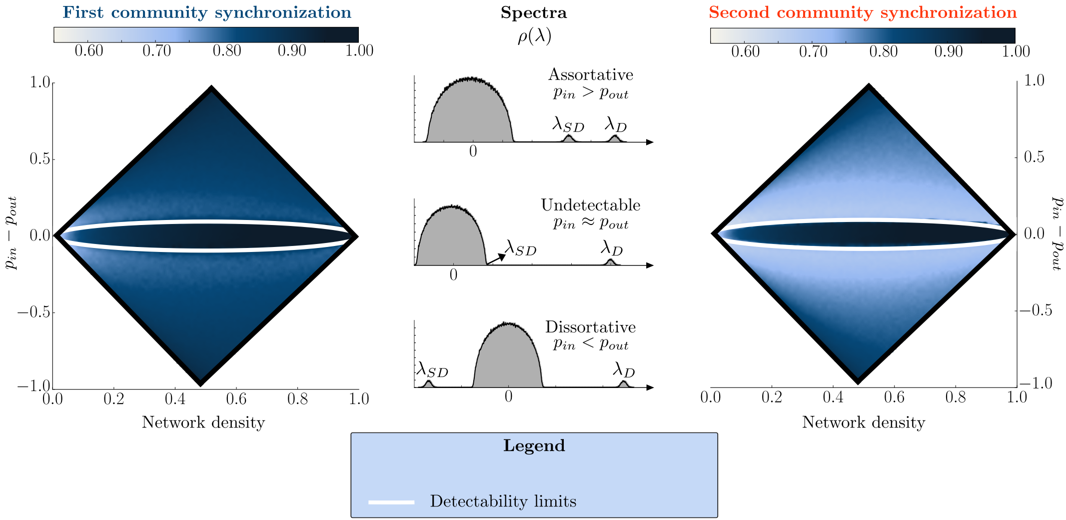

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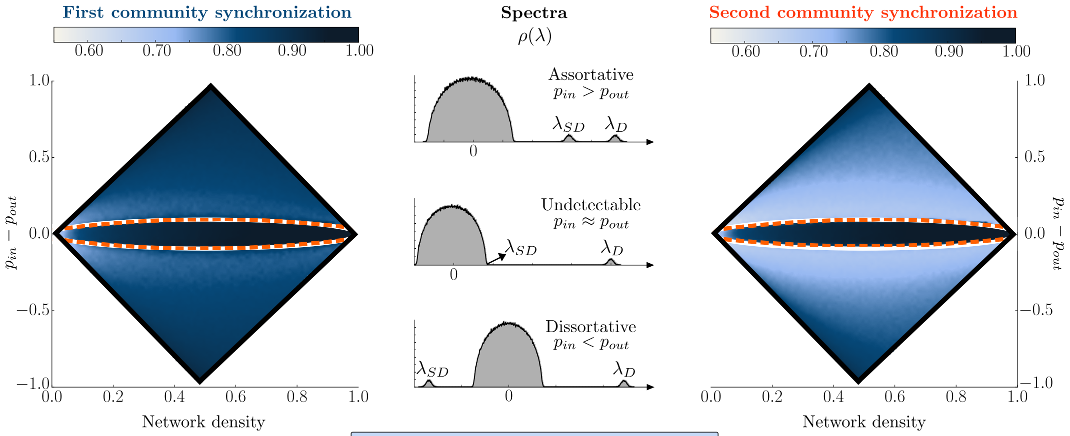

Legend

- Steepest synchronization variations
- Detectability limits

Summary

- Spectral graph theory allows to reduce successfully multiple synchronization dynamics

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- Detectability of the SBM delimits synchronization regions in the Cowan-Wilson dynamics

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Typical Netsci message : Structure influences the dynamics !

Acknowledgements

Thank you!

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Preprint : Coming soon

Contact : vincent.thibeault.1@ulaval.ca

Dimension reduction in synchronization

- Watanabe-Strogatz (1993)
- Ott-Antonsen (2008)
- **Spectral**(2018-2019) ***original approach***

Advantages of the spectral dimension reduction :

- $N < \infty$
- Systematic reduction of dynamics on graphs
- Few hypothesis
- Not restricted to synchronization dynamics