Dimension reduction of high-dimensional dynamics on networks with adaptation

Vincent Thibeault, Marina Vegué, Antoine Allard, and Patrick Desrosiers

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Département de physique, de génie physique, et d'optique
Université Laval, Québec, Canada

siam 2021 Conference on Applications of Dynamical Systems

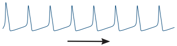
Dynamica

Université Laval

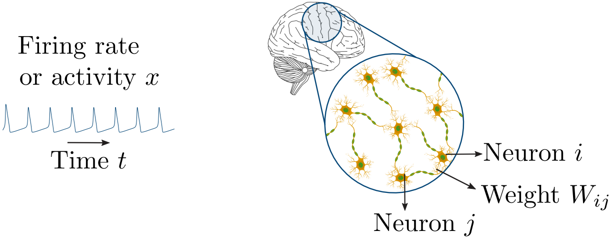
Emergence of collective phenomena (synchronization)

https://www.youtube.com/watch?v=tRPuVAVXk2M

Firing rate
or activity x



Time t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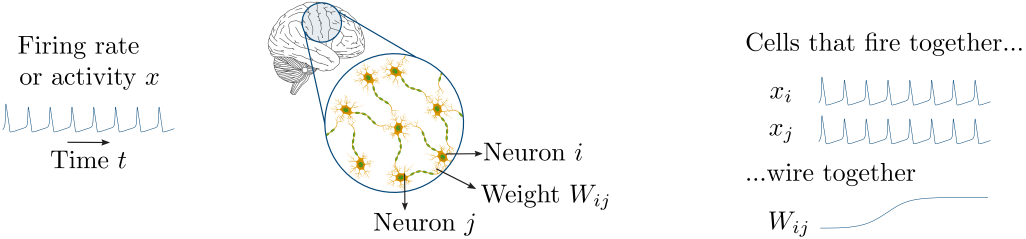
Neuron i
Weight W_{ij}
Neuron j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Cells that fire together...
x_i
x_j
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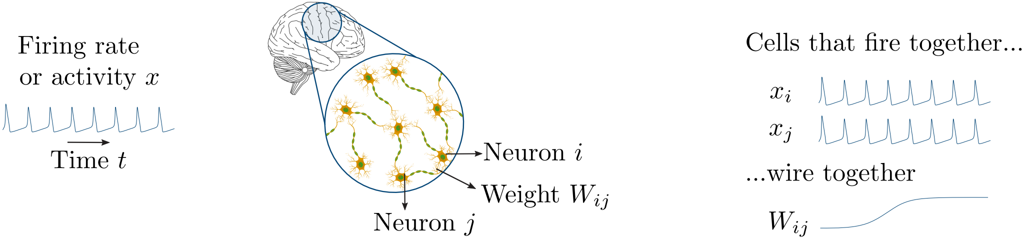
W_{ij}



Nonlinear activity dynamics + **Complex network** + **Nonlinear adaptation (plasticity)**

$$\frac{dx_i}{dt} = F(x_i) + G(x_i, \sum_{j=1}^{N} W_{ij} x_j)$$

$$\frac{dW_{ij}}{dt} = H(x_i, x_j, W_{ij})$$

Complete dynamics

$$N(N+1) \gg 1$$

$$\dot{x}_i = F(x_i) + G(x_i, \textstyle\sum_{j=1}^{N} W_{ij} x_j)$$
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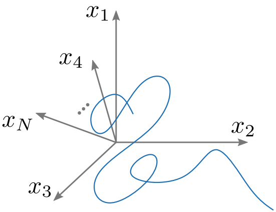

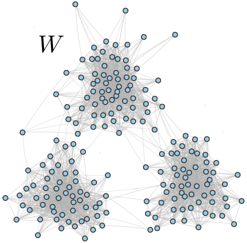

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$$\dot{\mathcal{X}}_\mu \approx ?$$

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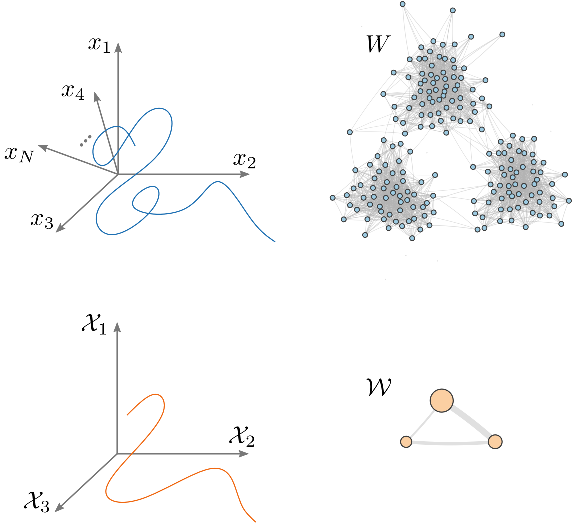

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Dimension reduction allows to ...

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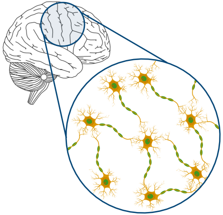

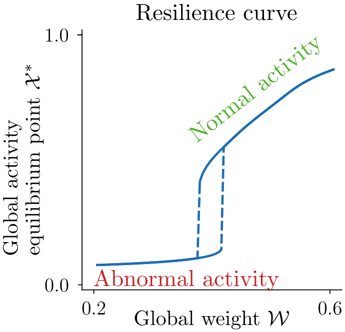

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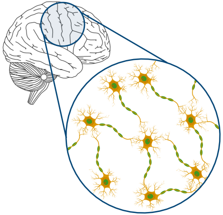

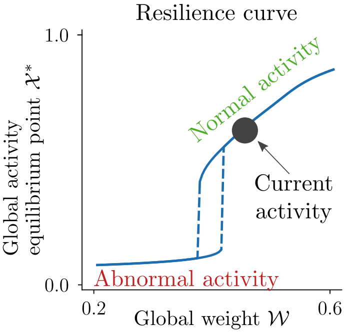

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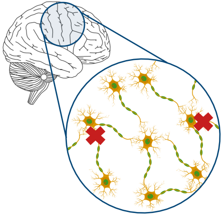

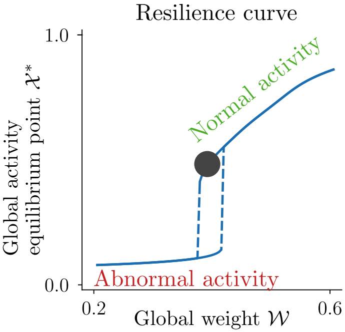

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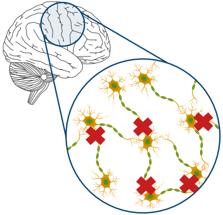

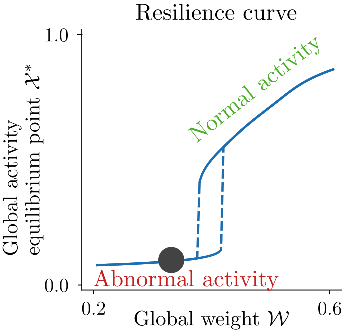

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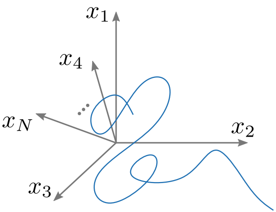

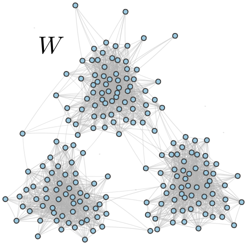

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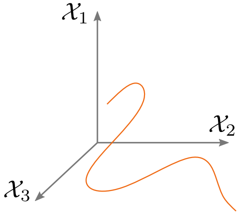

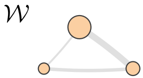

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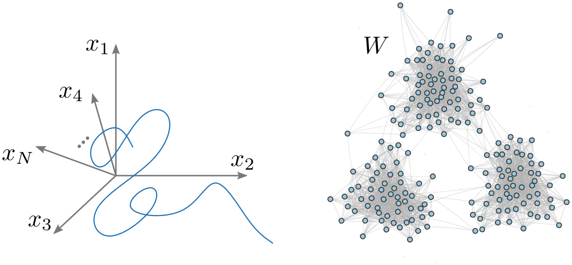

Reduced dynamics

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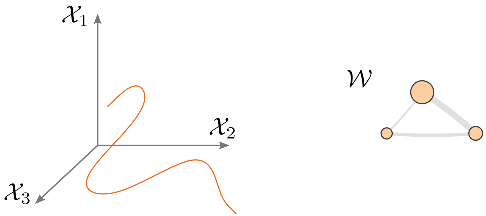

We found $n + n^2$ **linear observables (functions, measures,...)**

$$\mathcal{X}_\mu = \sum_{i=1}^{N} M_{\mu i} x_i,$$

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that both depend on only *one* $n \times N$ matrix M.

M is a *reduction matrix* **to be determined**.

Hypothesis

Important neurons contribute strongly to the global activity

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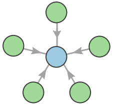
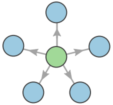
Example:
- Important paper
- Important review



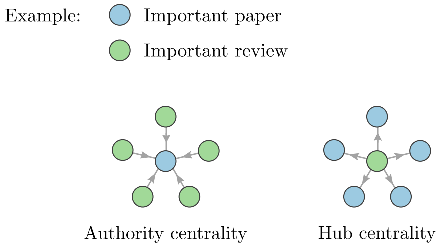
Authority centrality

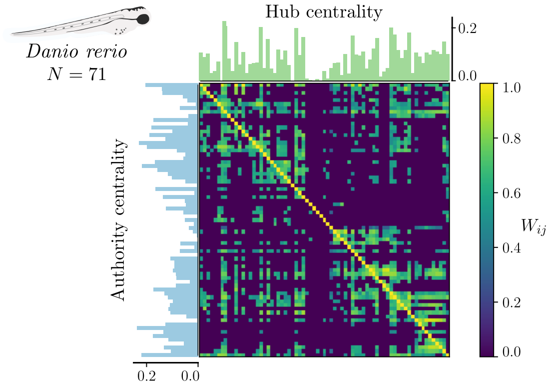


Hub centrality

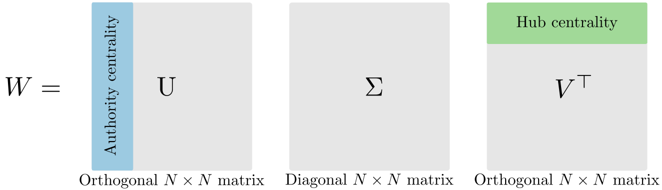
Hypothesis

Important neurons contribute strongly to the global activity





Singular value decomposition (SVD)



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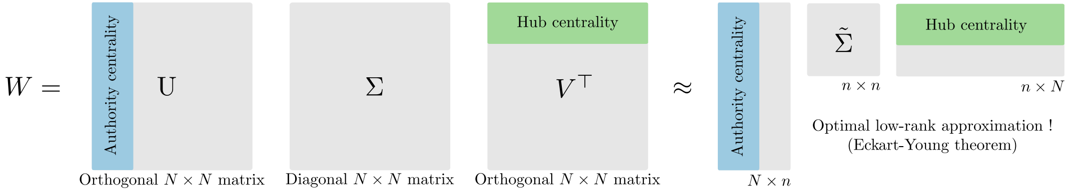

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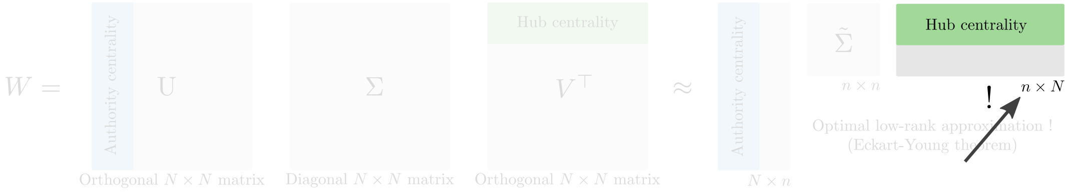

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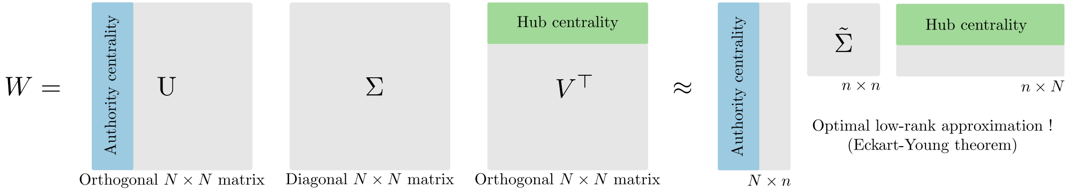
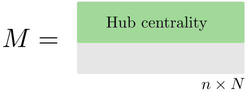


Reduction matrix



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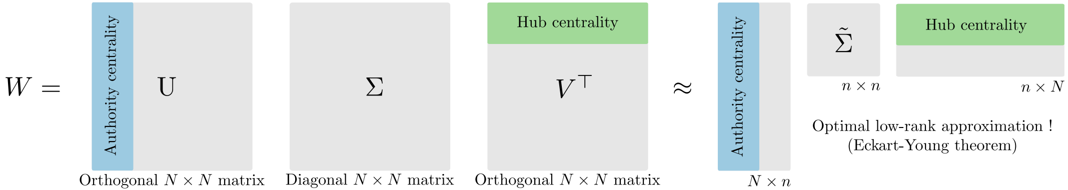

Optimal low-rank approximation !
(Eckart-Young theorem)

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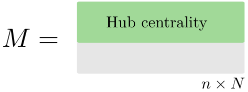

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$$\mathcal{X} = M\mathbf{x}$$

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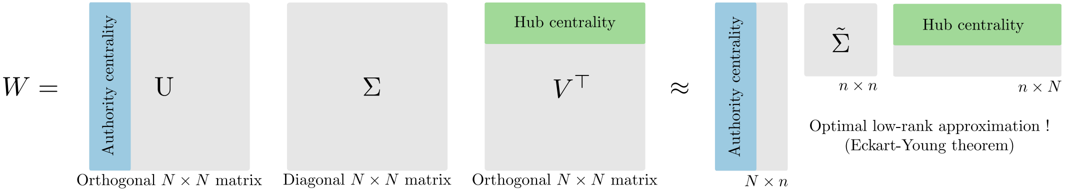

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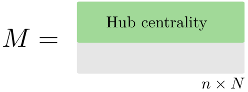

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3. Plot resilience curves $\mathcal{X}^*$ vs. $\mathcal{W}^*$.

Activity dynamics on a real network without plasticity

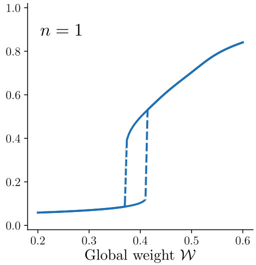
C. elegans
$N = 279$
$r = 273$

Activity dynamics on a real network without plasticity

y-axis

Global activity equilibrium point $\mathcal{X}^*$

Complete dynamics



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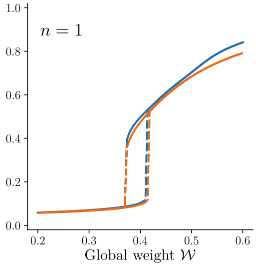

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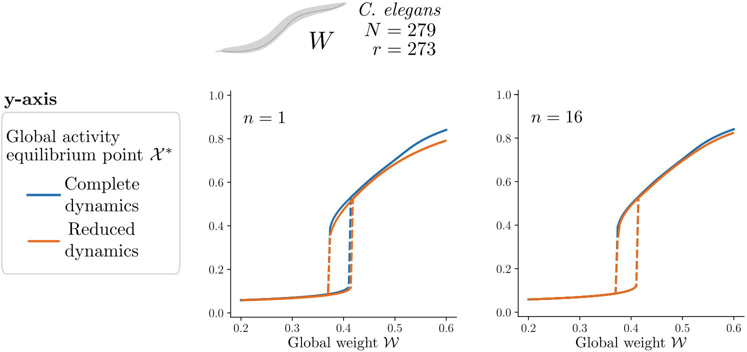

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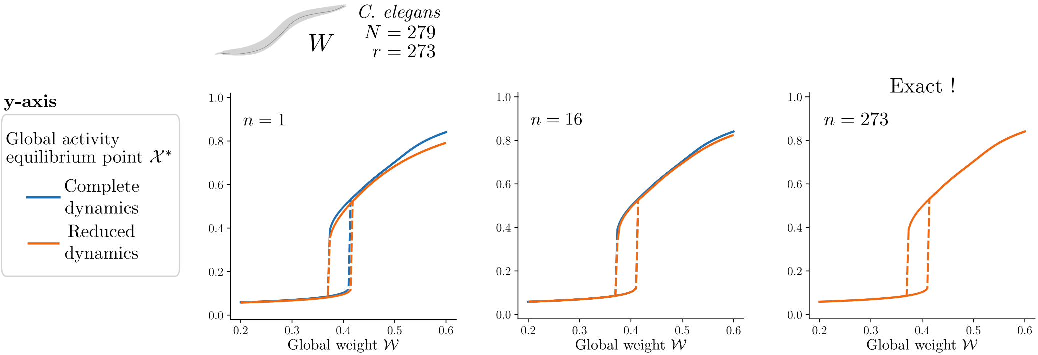

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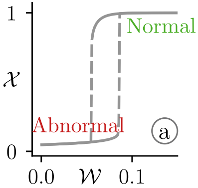

— No plasticity

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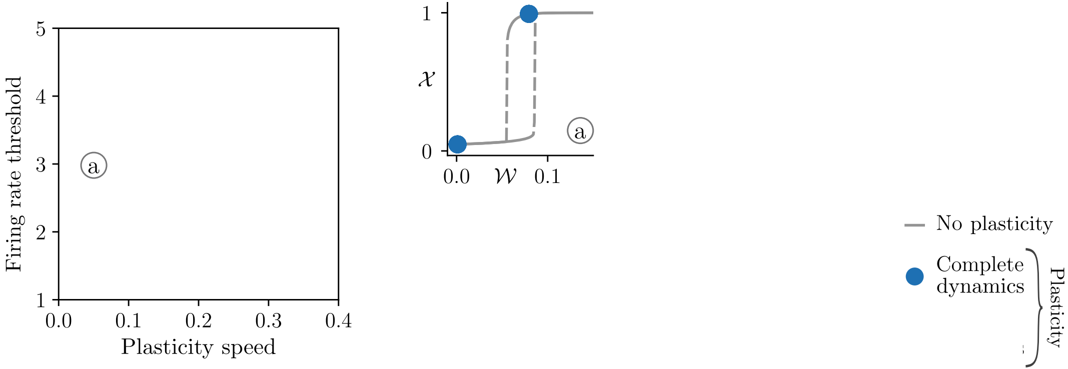

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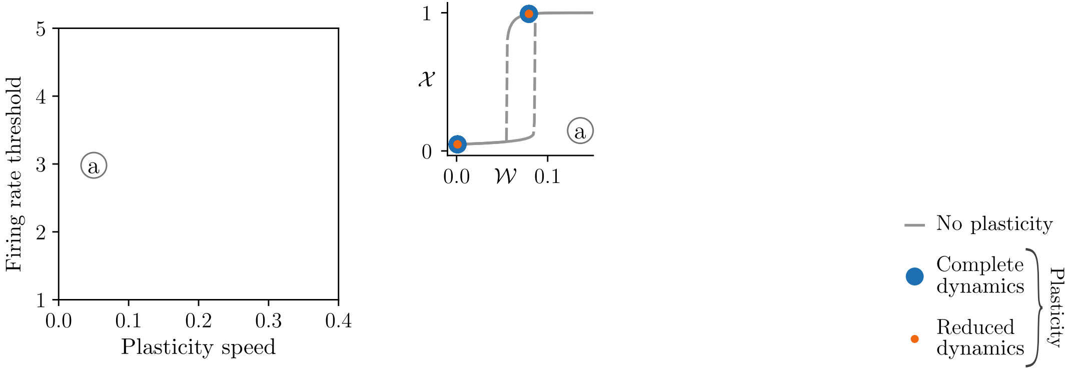

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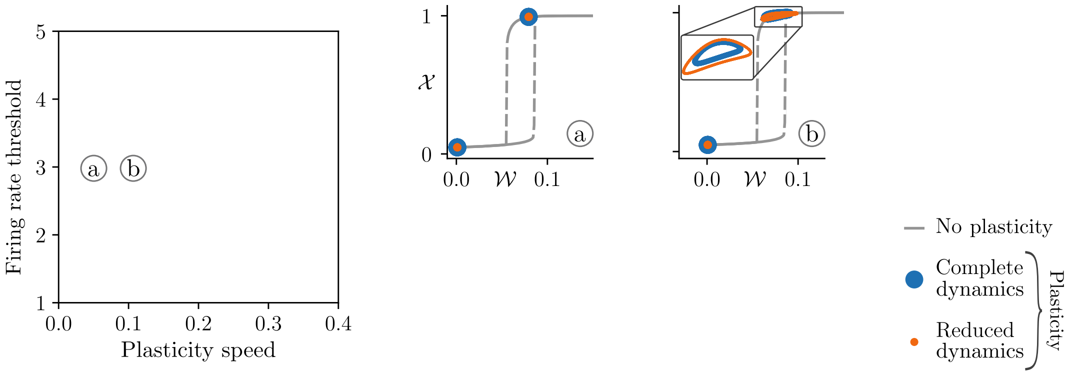

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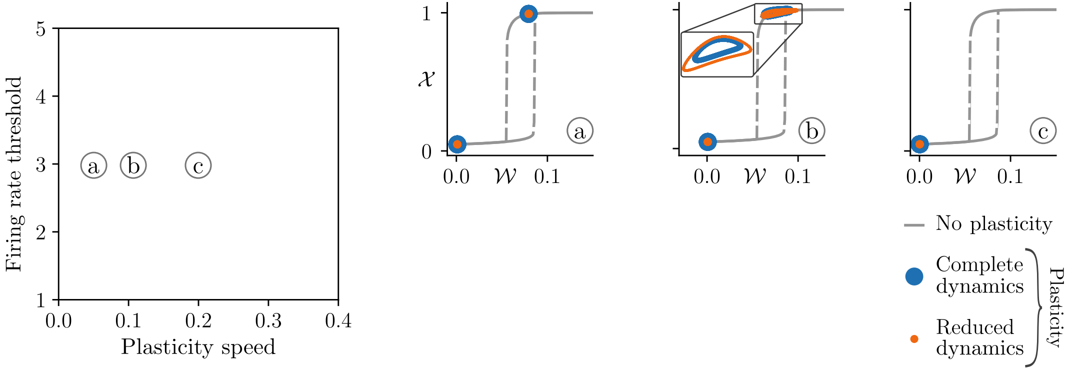

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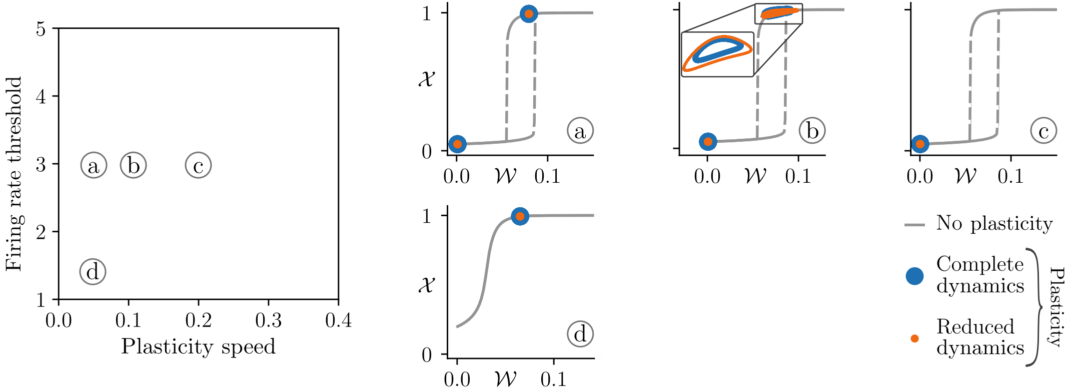

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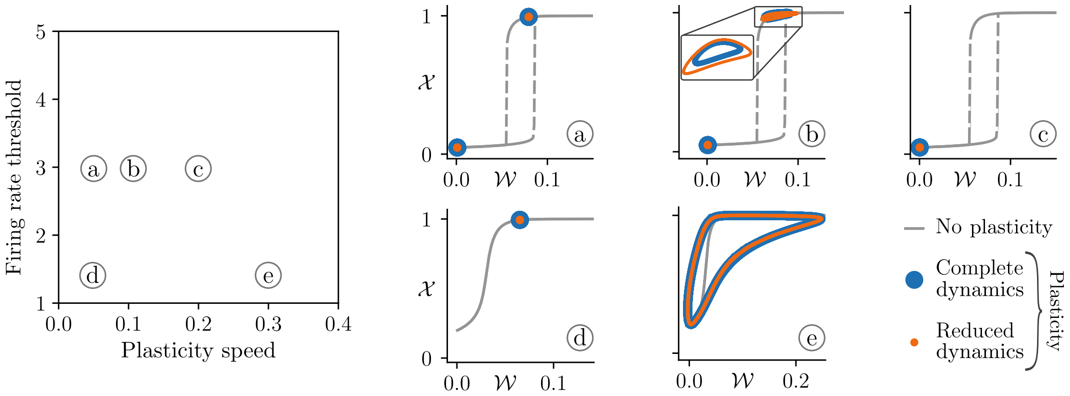

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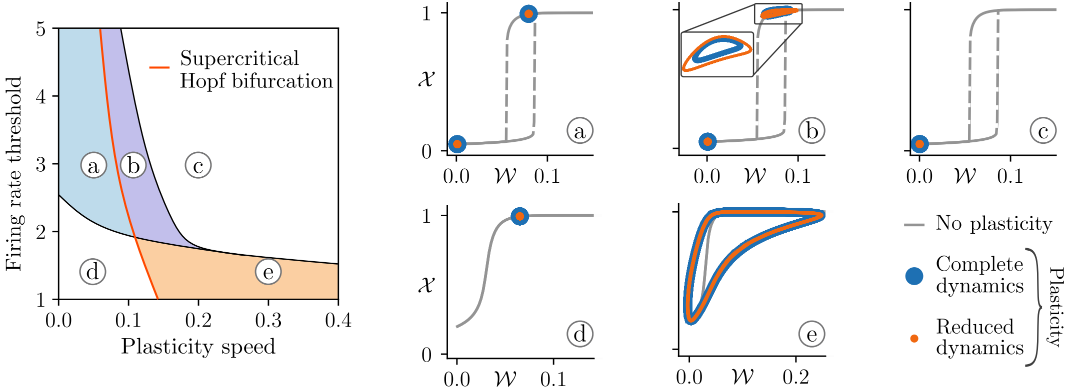

Next steps

- Treat plasticity + real networks ;
- Consider inhibitors ($W_{ij} < 0$) ;
- Use nonlinear observables ;
- Get more profound insights on resilience.

Take home messages

- Reduced dynamics are valuable to disentangle dynamics with plasticity ;
- SVD is a powerful and *interpretable* tool for dimension reduction *of dynamics*.

Thank you for your attention !

Thanks to the organizers !

Questions ?

SIAM 2021 Conference on Applications of Dynamical Systems

V. Thibeault et al., Phys. Rev. Res. (2020)

E. Laurence et al., Phys. Rev. X (2019)

J. Jiang et al., PNAS (2018)

J. Gao et al., Nature (2016)

Coauthors : M.Vegué, A. Allard, P. Desrosiers

Contact : vincent.thibeault.1@ulaval.ca

Website : `https://dynamicalab.github.io/`

CIMMUL Centre Interdisciplinaire en Modélisation Mathématique de l'Université Laval

Dynamica

CRSNG NSERC

Fonds de recherche Nature et technologies Québec

Université Laval

Centre de recherche CERVO Brain Research Centre Québec

Sentinelle Nord

Calcul Québec

In this model, F is linear and G is a sigmoid function :

$$\tau_x \, \dot{x}_i = -x_i + 1/(1 + e^{-a(y_i - b)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j \tag{1}$$

- x_i : Firing rate of neuron or brain region i
- τ_x : Time scale of the firing rate
- a : Steepness of the activation function
- b : Firing rate threshold

The Wilson-Cowan model is described by the set of differential equations

$$\dot{x}_i = -\alpha x_i + G(\textstyle\sum_{j=1}^{N} W_{ij} x_j), \quad i \in \{1, ..., N\},$$

where G is the sigmoid function. By defining $x = (x_1 \quad \ldots \quad x_N)^\top$, we have the equivalent form

$$\dot{x} = -\alpha x + G(Wx). \tag{2}$$

The reduced dynamics for $X = Mx$ is

$$\dot{X} = -\alpha X + MG(LX), \tag{3}$$

where we have rank-factorized W as LM.

This model is more complex :

$$\tau_x\,\dot{x}_i = -\alpha_i x_i + \beta_i/(1+e^{-a(y_i-b)}), \quad \text{with} \quad y_i = \textstyle\sum_{j=1}^{N} W_{ij}x_j + \gamma_i \tag{4}$$

$$\tau_w\,\dot{W}_{ij} = D_{ij}x_i x_j(x_i-\theta_i) - \varepsilon W_{ij} \quad \text{with} \quad W_{ij}(0) = d_{ij}D_{ij} \tag{5}$$

$$\tau_\theta\,\dot{\theta}_i = x_i^2 - \theta_i. \tag{6}$$

θ_i : modify the threshold above (below) which the synapse potentiates (depresses).

$\alpha_i, \beta_i, \gamma_i$: distinguish the dynamical behavior of each node i.

$D = (D_{ij})_{i,j=1}^{N}$: structural backbone, $D_{ij} > 0$ if the presynaptic neuron j excites the postsynaptic neuron i, $D_{ij} < 0$ if the presynaptic neuron j inhibits the postsynaptic neuron i, and $D_{ij} = 0$ if no edge exist between neurons i and j.

The reduced dynamics is described by the differential equations

$$\dot{\mathcal{X}}_\mu \approx F(\mathcal{X}_\mu; \alpha_\mu) + G(\mathcal{X}_\mu, \mathcal{Y}_\mu; \beta_\mu) \quad \text{with} \quad \mathcal{Y}_\mu = \textstyle\sum_{\rho=1}^{n} \mathcal{W}_{\mu\rho}\mathcal{X}_\rho + \gamma_\mu \tag{7}$$

$$\dot{\mathcal{W}}_{\mu\nu} \approx \mathcal{D}_{\mu\nu} H(\mathcal{X}_\mu, \mathcal{X}_\nu, \Theta_\mu) - \mathcal{W}_{\mu\nu} J(\mathcal{X}_\mu, \mathcal{X}_\nu) \tag{8}$$

$$\dot{\Theta}_\mu \approx T(\mathcal{X}_\mu, \Theta_\mu) \tag{9}$$

where

- $\xi_\mu = \sum_i \hat{M}_{\mu i}\xi_i$ with $\xi \in \{\alpha, \beta, \gamma\}$
- $\mathcal{D}_{\mu\nu} = \sum_{i,j=1}^{N} M_{\mu i} D_{ij} M_{j\nu}^\top$
- $\mathcal{W}_{\mu\nu}(0) = \mathcal{D}_{\mu\nu}$ for all $\mu, \nu \in \{1, ..., n\}$