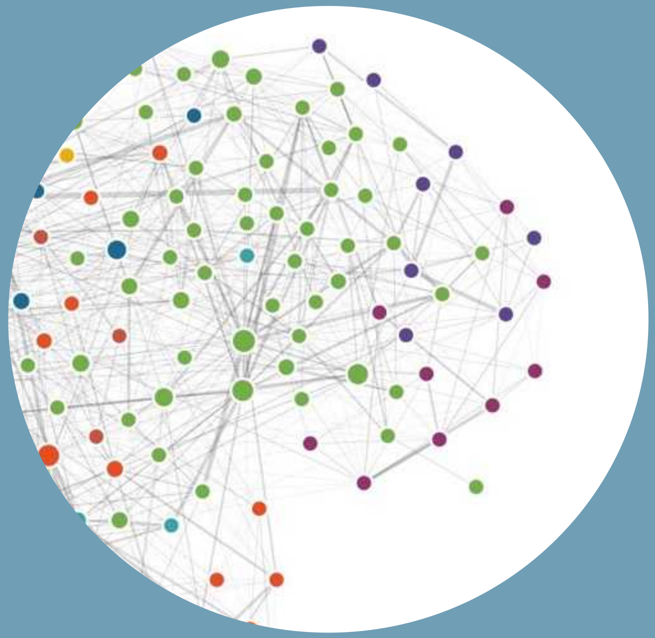
Dimension reduction on heterogeneous networks



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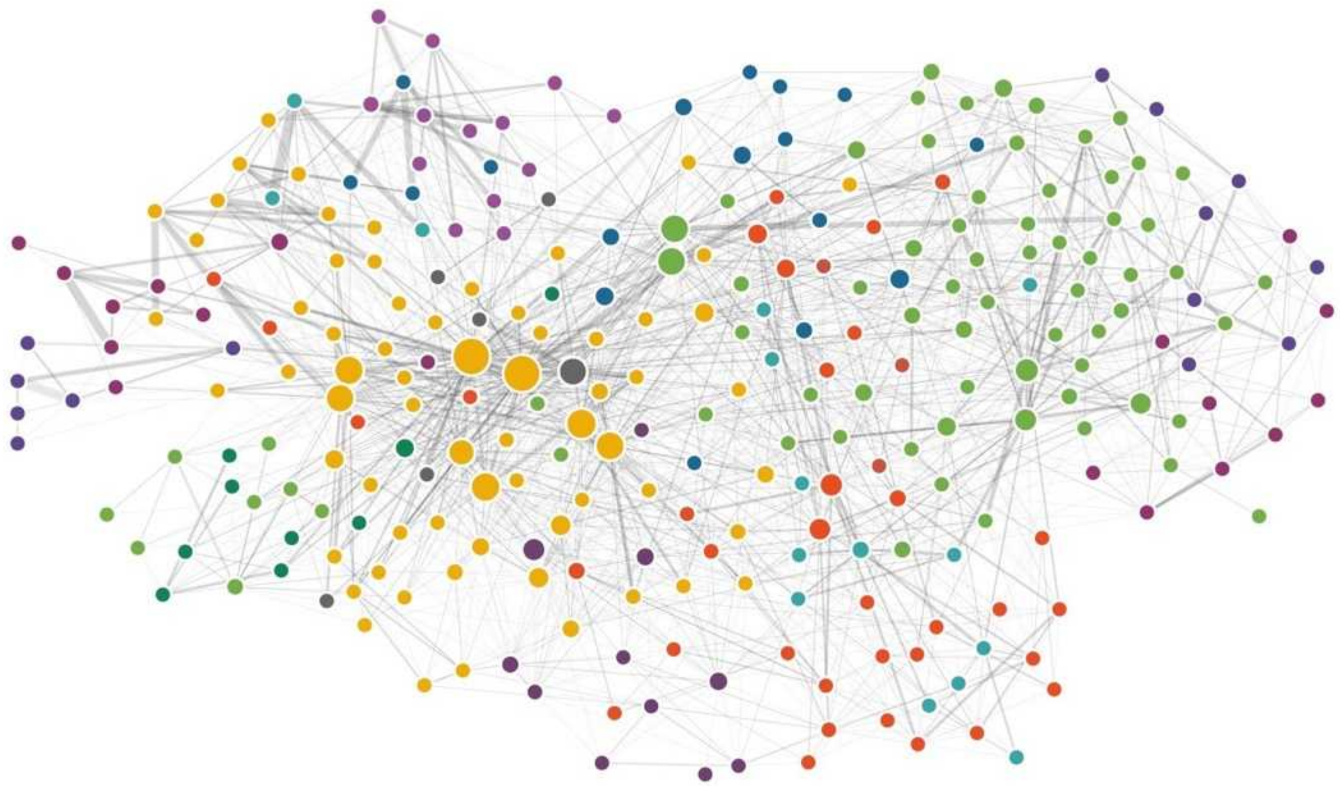
Why dimension reduction?



Goal

Find a network of reduced size whose dynamics can be used to infer some basic properties of the original, high dimensional, dynamics.

Use it to study systems whose units exhibit **non-symmetric** and **heterogeneous interactions.**

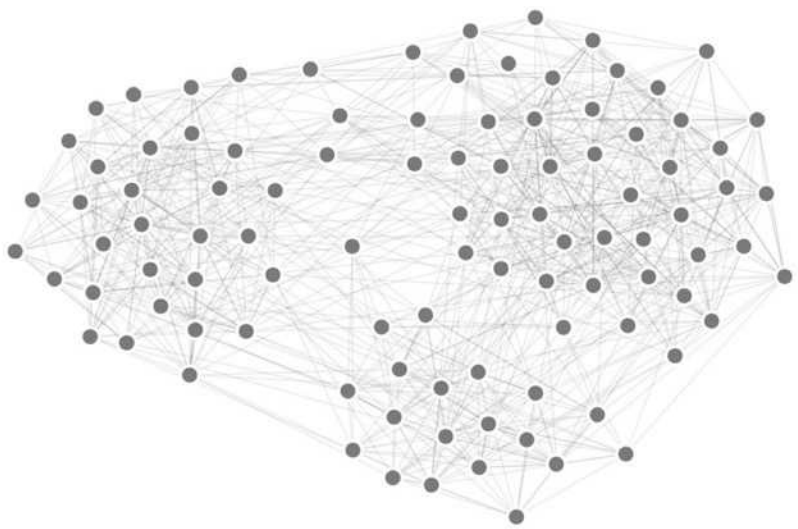
Previous work

Gao et al., Nature, 2016
Jiang et al., PNAS, 2018
Laurence et al., Phys. Rev. X, 2019
Thibeault et al., iScience, 2020

Original

N nodes

Network



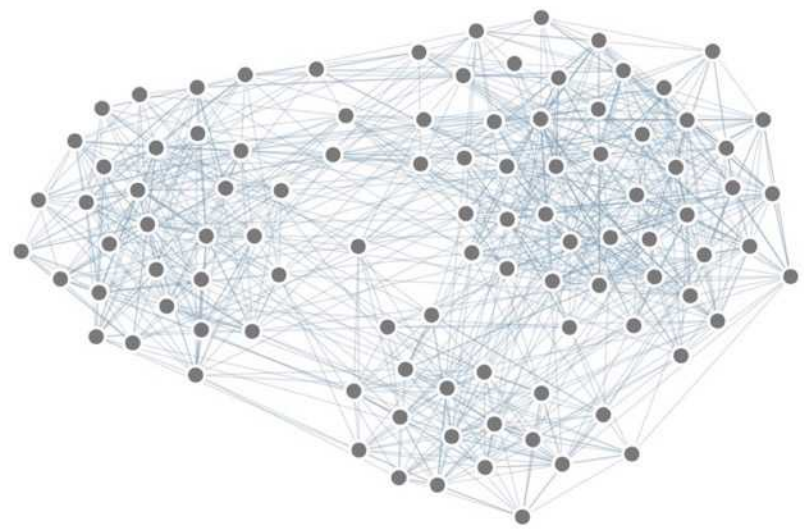
Dynamics

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} w_{ij}\, g(x_i, x_j)$$

Original

N nodes

Network



$j \xrightarrow{\boldsymbol{w}_{ij}} i$

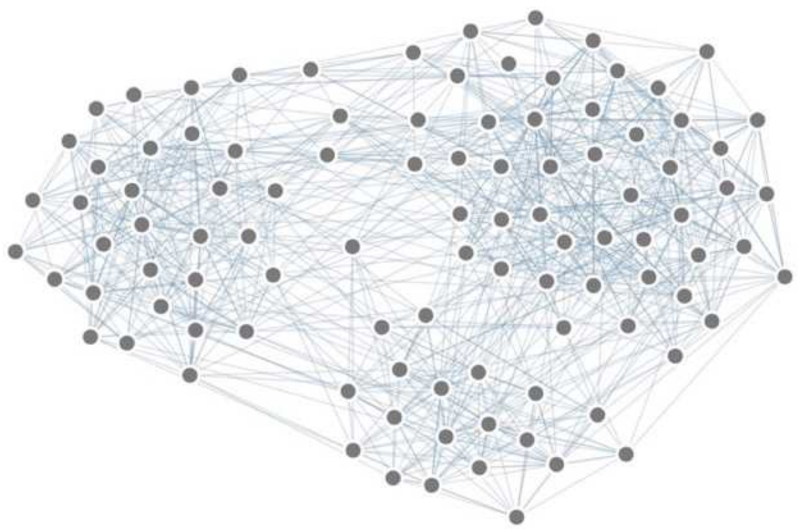
Dynamics

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} \boldsymbol{w}_{ij}\, g(x_i, x_j)$$

Original

N nodes

Network



Dynamics

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} \boldsymbol{w}_{ij}\, g(x_i, x_j)$$

$$f(x) = -x$$

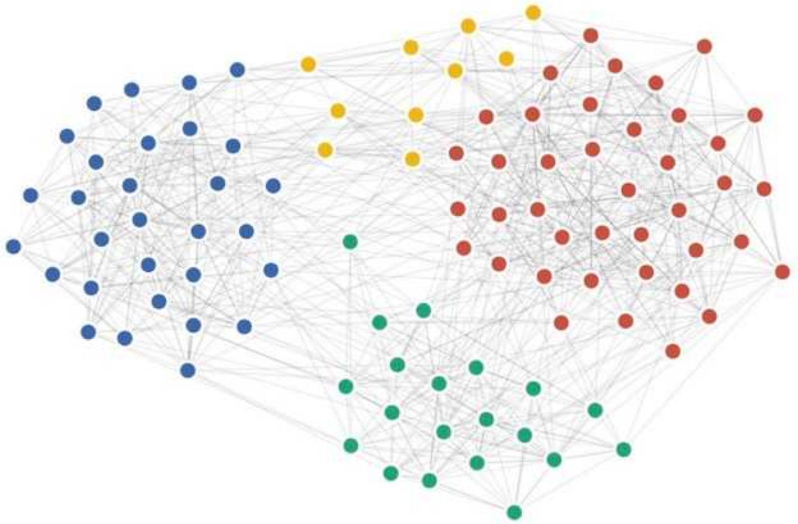
$$g(x, y) = \frac{1}{1+\exp\left(-\tau(y-\mu)\right)}$$

Additive model
(Hopfield, PNAS, 1984)

Original

N nodes

Network



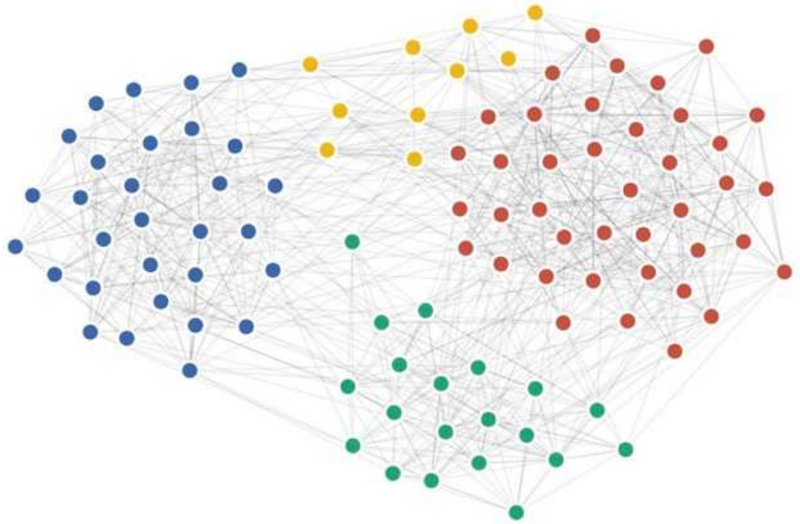
Dynamics $\dot{x}_i = f(x_i) + \sum_{j=1}^{N} w_{ij}\, g(x_i, x_j)$

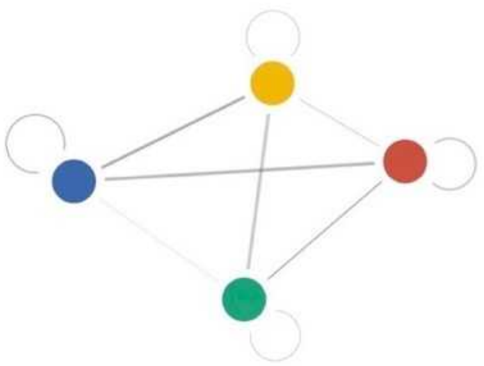
Steps 1. Community / group detection

Original | **Reduced**

N nodes | n nodes

Network



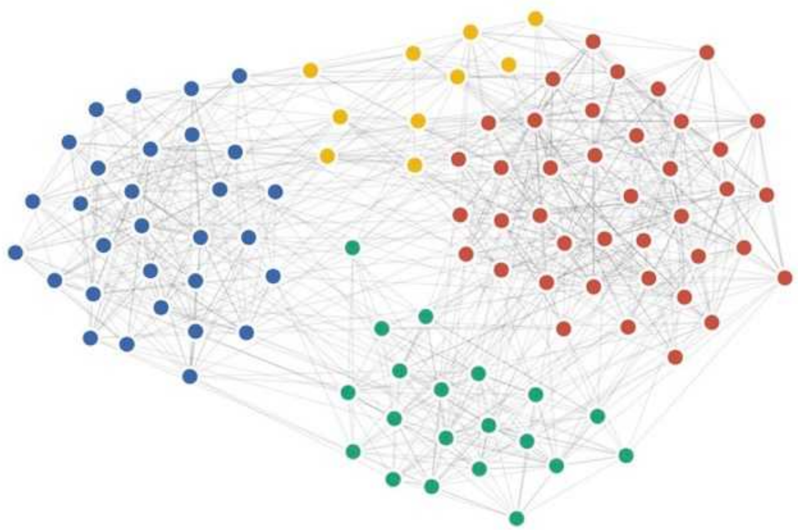
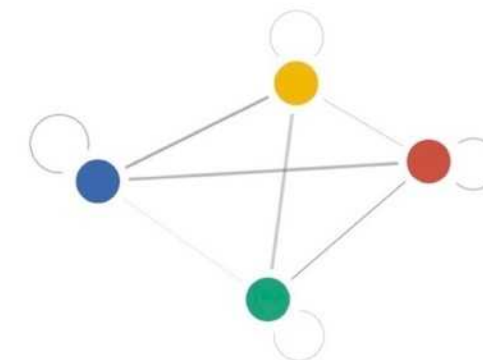


Dynamics

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} w_{ij}\, g(x_i, x_j)$$

Steps

1. Community / group detection

	Original	Reduced
	N nodes	n nodes
Network		
Dynamics	$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} w_{ij}\, g(x_i, x_j)$	$\dot{\mathcal{X}}_\nu = f(\mathcal{X}_\nu) + \sum_{\rho=1}^{n} \mathcal{W}_{\nu\rho}\, g(\mathcal{X}_\nu, \mathcal{X}_\rho)$

Steps

1. Community / group detection
2. Define $\{\mathcal{X}_\nu, \mathcal{W}_{\nu\rho}\}_{\nu,\rho}$ from $\{x_i, w_{ij}\}_{i,j}$

1. **Observables are linear combinations of the node activities within each group**

$$\mathcal{X}_\nu = \sum_{i=1}^{N} [\boldsymbol{a_\nu}]_i \, x_i, \quad [\boldsymbol{a_\nu}]_i = 0 \text{ if } i \notin G_\nu, \quad \sum_{i=1}^{N} [\boldsymbol{a_\nu}]_i = 1$$

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Exact observable dynamics

$$\dot{\mathcal{X}}_\nu = \sum_{i=1}^{N} [\boldsymbol{a}_\nu]_i f(x_i) + \sum_{i,j=1}^{N} [\boldsymbol{a}_\nu]_i \, w_{ij} \, g(x_i, x_j)$$

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2. **Assume that the activity of each node is *close enough* to the corresponding observable**

$$x_i \approx \mathcal{X}_\nu \text{ for } i \in G_\nu$$

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2. Assume that the activity of each node is *close enough* to the corresponding observable

$x_i \approx \mathcal{X}_\nu$ for $i \in G_\nu$

3. For $\boldsymbol{i \in G_\nu, j \in G_\rho}$, approximate

a) $f(x_i) \approx f(\mathcal{X}_\nu)$, $g(x_i, x_j) \approx g(\mathcal{X}_\nu, \mathcal{X}_\rho)$

The observable dynamics becomes closed without imposing any additional condition on $\{\boldsymbol{a_\nu}\}_\nu$

1.	**Observables are linear combinations of the node activities within each group**	$\mathcal{X}_\nu = \sum_{i=1}^{N} [\boldsymbol{a_\nu}]_i \, x_i, \quad [\boldsymbol{a_\nu}]_i = 0 \text{ if } i \notin G_\nu, \quad \sum_{i=1}^{N} [\boldsymbol{a_\nu}]_i = 1$
	Exact observable dynamics	$\dot{\mathcal{X}}_\nu = \sum_{i=1}^{N} [\boldsymbol{a}_\nu]_i f(x_i) + \sum_{i,j=1}^{N} [\boldsymbol{a}_\nu]_i \, w_{ij} \, g(x_i, x_j)$
2.	**Assume that the activity of each node is *close enough* to the corresponding observable**	$x_i \approx \mathcal{X}_\nu$ for $i \in G_\nu$
3.	**For $\boldsymbol{i \in G_\nu, j \in G_\rho}$, approximate**	
	a) $f(x_i) \approx f(\mathcal{X}_\nu)$, $g(x_i, x_j) \approx g(\mathcal{X}_\nu, \mathcal{X}_\rho)$	The observable dynamics becomes closed without imposing any additional condition on $\{\boldsymbol{a_\nu}\}_\nu$
	b) $f(x_i), g(x_i, x_j)$ by 1st-order Taylor polynomials around $\mathcal{X}_\nu, (\mathcal{X}_\nu, \mathcal{X}_\rho)$	Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

a) The observable dynamics becomes closed without imposing any additional condition on $\{\boldsymbol{a_\nu}\}_\nu$

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$$[\boldsymbol{a_\nu}]_i = \begin{cases} 1/|G_\nu| & i \in G_\nu \\ 0 & i \notin G_\nu \end{cases} \qquad\qquad \mathcal{W}_{\nu\rho} = \frac{1}{|G_\nu|} \sum_{i \in G_\nu} \sum_{j \in G_\rho} w_{ij}$$

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Homogeneous reduction

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

$$\boldsymbol{a_\nu} = (0, \cdots, 0, \overbrace{*, \cdots, *}^{\widehat{\boldsymbol{a}}_\nu}, 0, \cdots, 0)^T$$

$\boldsymbol{W}_{\nu\rho}$
Interaction matrix from nodes in G_ρ to nodes in G_ν

$\boldsymbol{K}_{\nu\rho}$
Diagonal in-degree matrix of nodes in G_ν for interactions coming from G_ρ

a) The observable dynamics becomes closed without imposing any additional condition on $\{\boldsymbol{a_\nu}\}_\nu$

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

$$\boldsymbol{W}_{\boldsymbol{\nu\rho}}^{\boldsymbol{T}} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}} = \mathcal{W}_{\nu\rho} \widehat{\boldsymbol{a}}_{\boldsymbol{\rho}} \qquad \boldsymbol{K}_{\boldsymbol{\nu\rho}} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}} = \mathcal{W}_{\nu\rho} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}}$$

$$\boldsymbol{a_\nu} = (0, \cdots, 0, \overbrace{*, \cdots, *}^{\widehat{\boldsymbol{a}}_\nu}, 0, \cdots, 0)^T$$

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

$$\boldsymbol{W^T_{\nu\rho}} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}} = \mathcal{W}_{\nu\rho} \widehat{\boldsymbol{a}}_{\boldsymbol{\rho}} \qquad \boldsymbol{K_{\nu\rho}} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}} = \mathcal{W}_{\nu\rho} \widehat{\boldsymbol{a}}_{\boldsymbol{\nu}}$$

Compatibility equations

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

$$\boldsymbol{W^T_{\nu\rho}} \boldsymbol{\widehat{a}_\nu} = \mathcal{W}_{\nu\rho} \boldsymbol{\widehat{a}_\rho} \qquad \boldsymbol{K_{\nu\rho}} \boldsymbol{\widehat{a}_\nu} = \mathcal{W}_{\nu\rho} \boldsymbol{\widehat{a}_\nu}$$

Compatibility equations

Spectral reduction

a) The observable dynamics becomes closed without imposing any additional condition on $\{\boldsymbol{a_\nu}\}_\nu$

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Homogeneous reduction

b) Some conditions have to be imposed on $\{\boldsymbol{a_\nu}\}_\nu$ to close the observable dynamics

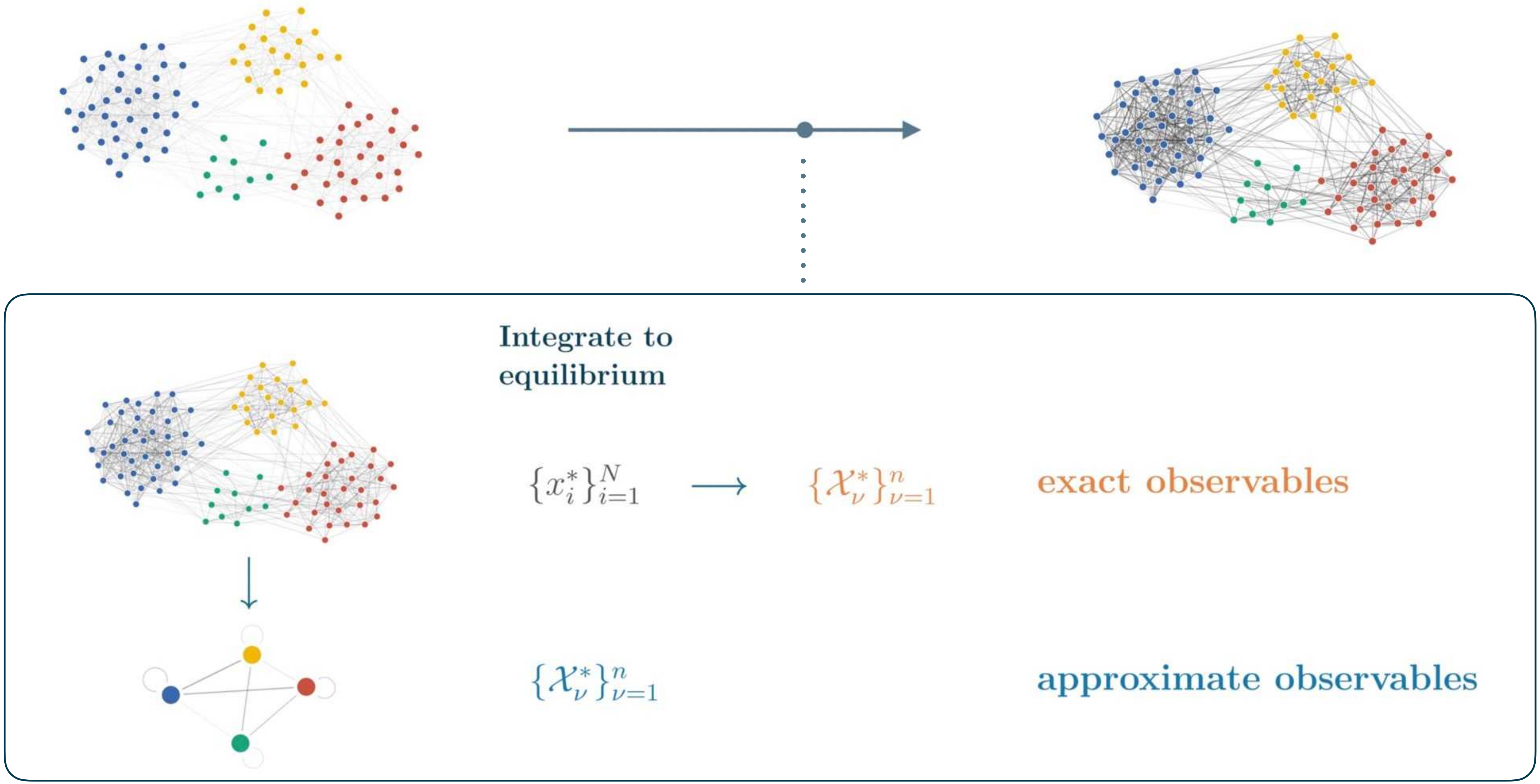
$$\boldsymbol{W_{\nu\rho}^T} \boldsymbol{\widehat{a}_\nu} = \mathcal{W}_{\nu\rho} \boldsymbol{\widehat{a}_\rho} \qquad \boldsymbol{K_{\nu\rho}} \boldsymbol{\widehat{a}_\nu} = \mathcal{W}_{\nu\rho} \boldsymbol{\widehat{a}_\nu}$$

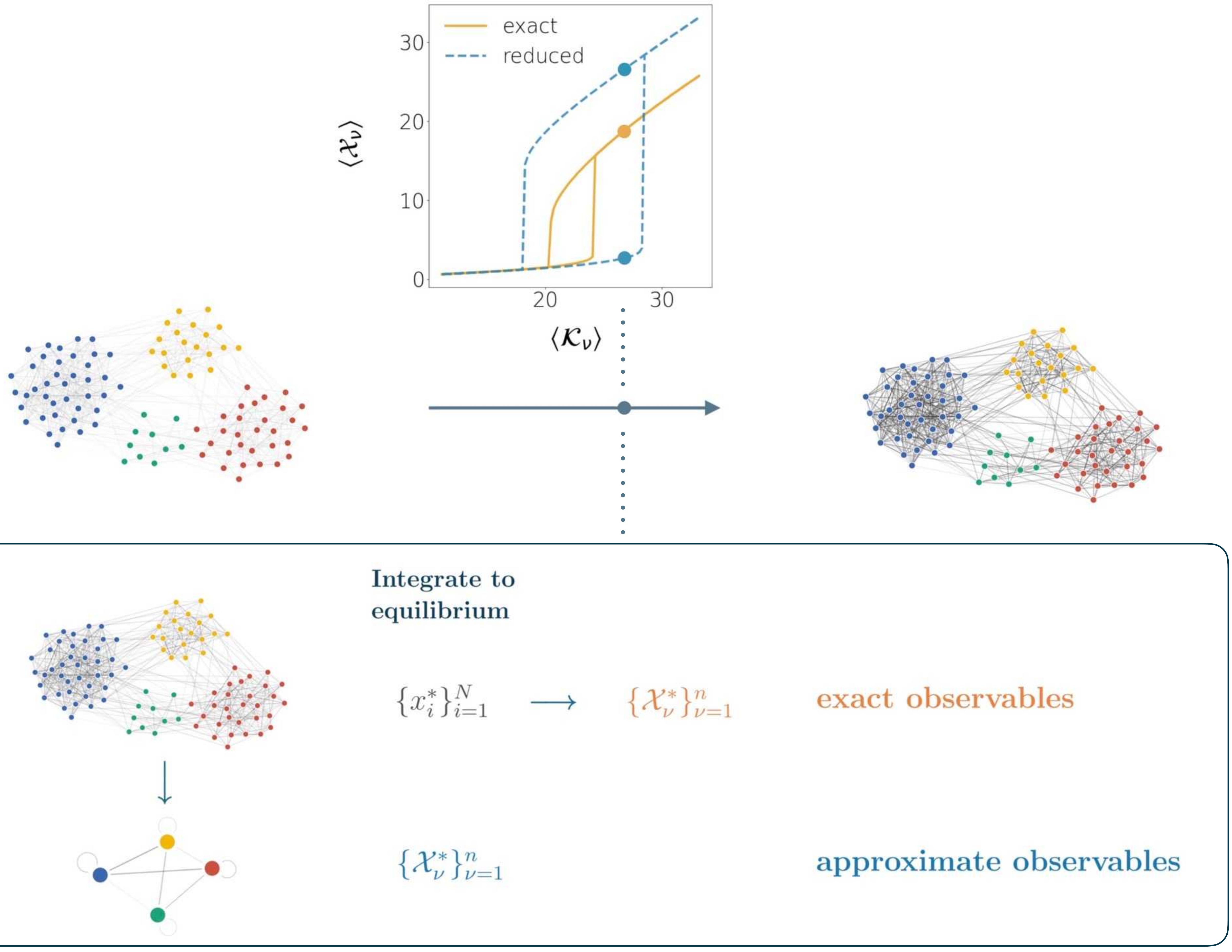
Compatibility equations

Spectral reduction

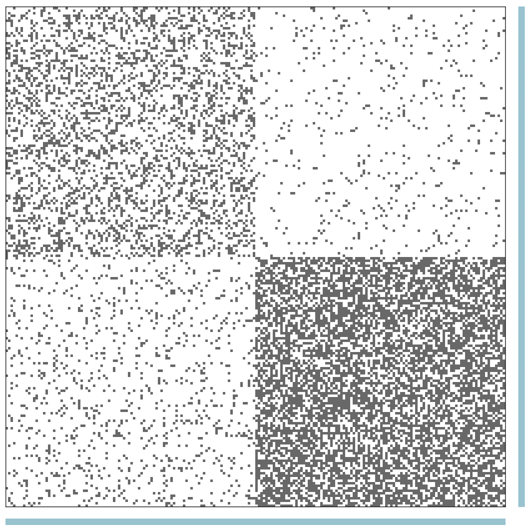
$$\dot{\mathcal{X}}_\nu = f(\mathcal{X}_\nu) + \sum_{\rho=1}^{n} \mathcal{W}_{\nu\rho}\, g(\mathcal{X}_\nu, \mathcal{X}_\rho)$$

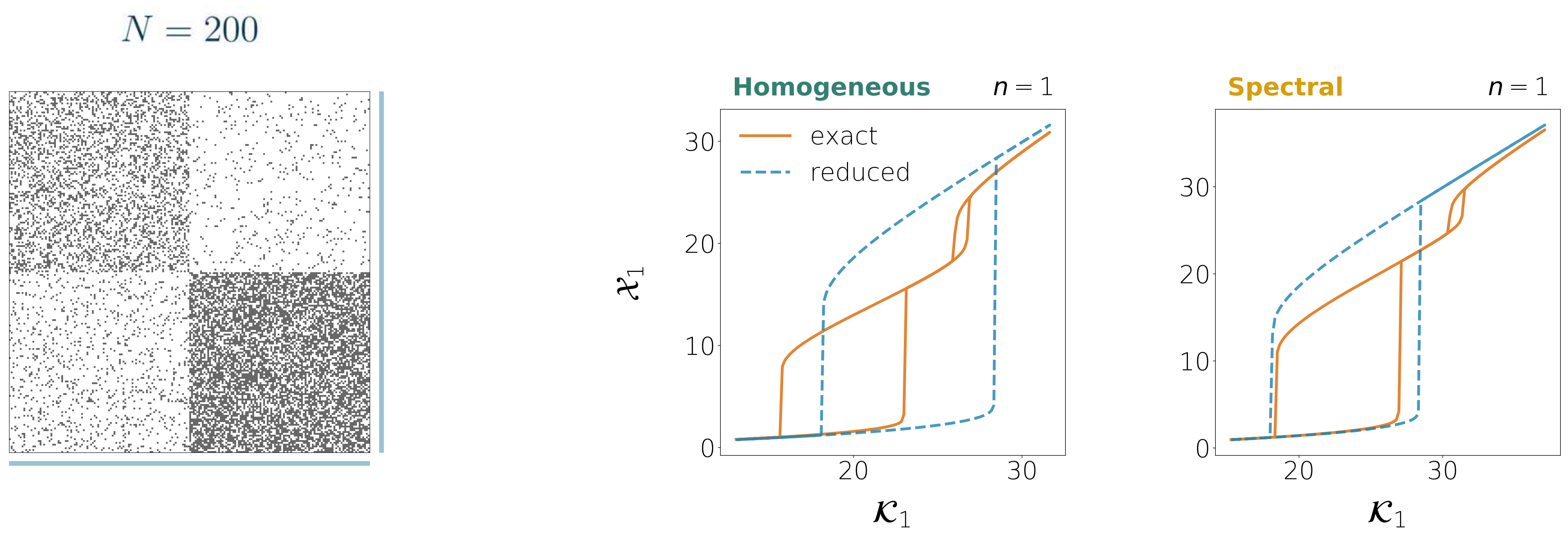
Approximate reduced dynamics


Integrate to equilibrium
$\{x_i^*\}_{i=1}^N \longrightarrow \{\mathcal{X}_\nu^*\}_{\nu=1}^n$
exact observables
$\{\mathcal{X}_\nu^*\}_{\nu=1}^n$
approximate observables

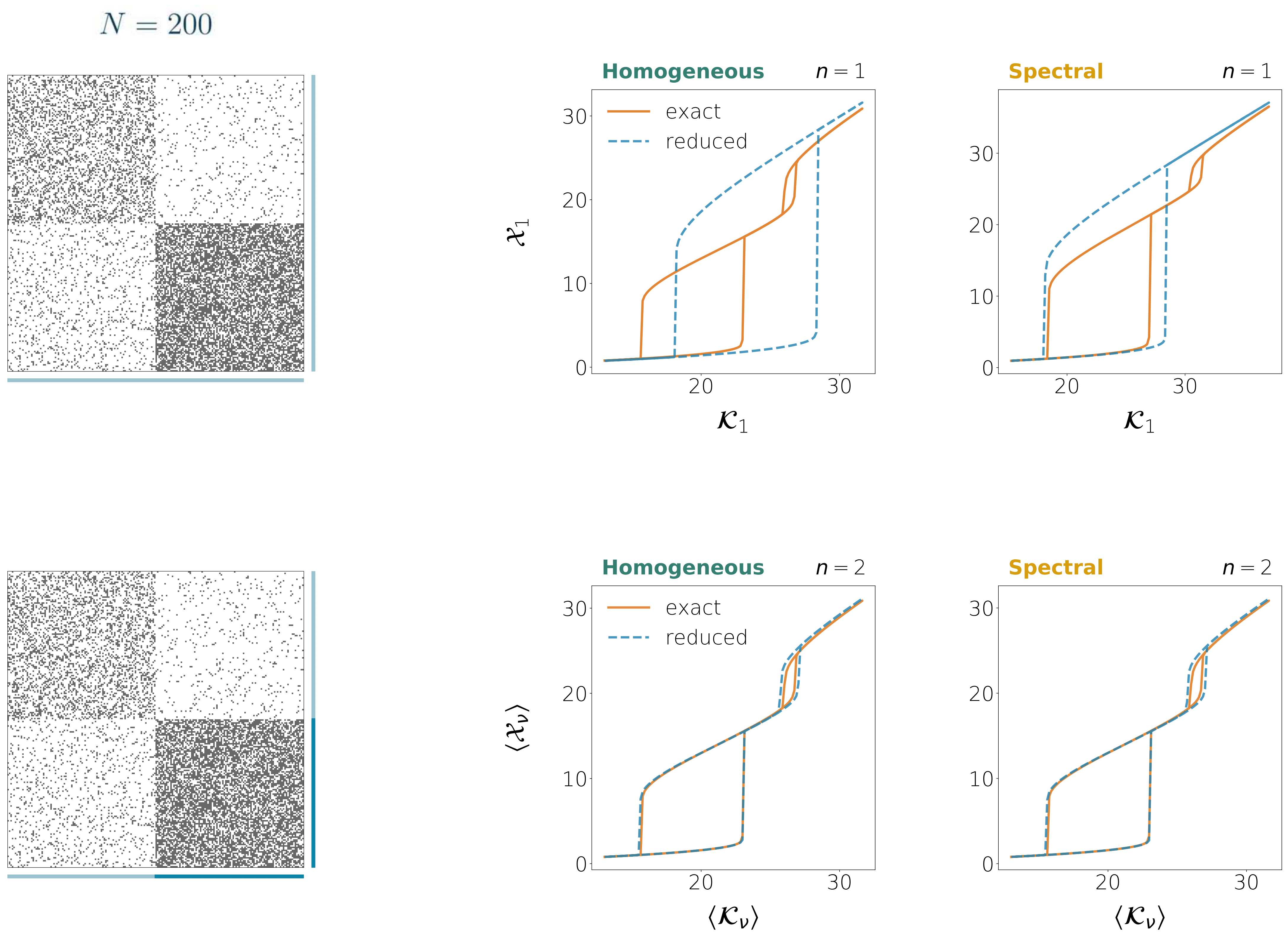

exact
reduced
30
20
10
0
20
30
$\langle \mathcal{X}_\nu \rangle$
$\langle \mathcal{K}_\nu \rangle$
Integrate to equilibrium
$\{x_i^*\}_{i=1}^N \longrightarrow \{\mathcal{X}_\nu^*\}_{\nu=1}^n$
exact observables
$\{\mathcal{X}_\nu^*\}_{\nu=1}^n$
approximate observables

$N = 200$



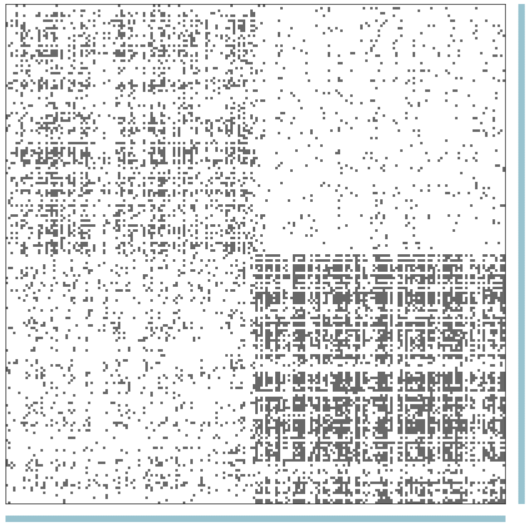


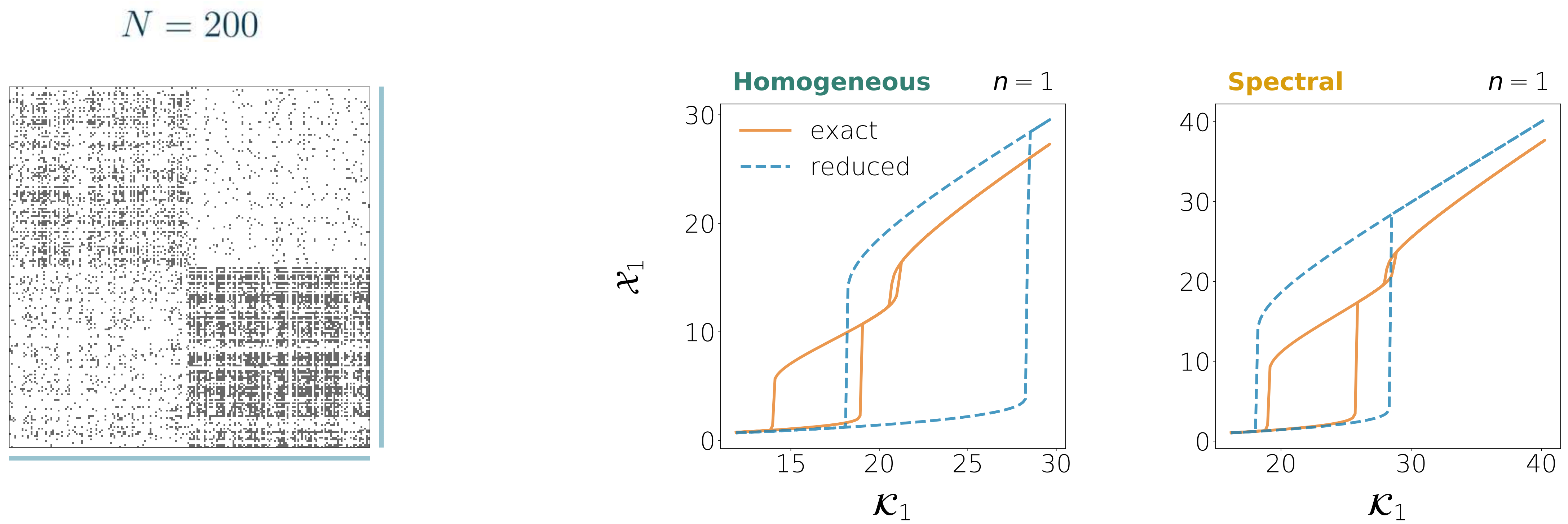
$N = 200$
Homogeneous
$n = 1$
exact
reduced
$\mathcal{X}_1$
$\mathcal{K}_1$
Spectral
$n = 1$
$\mathcal{K}_1$



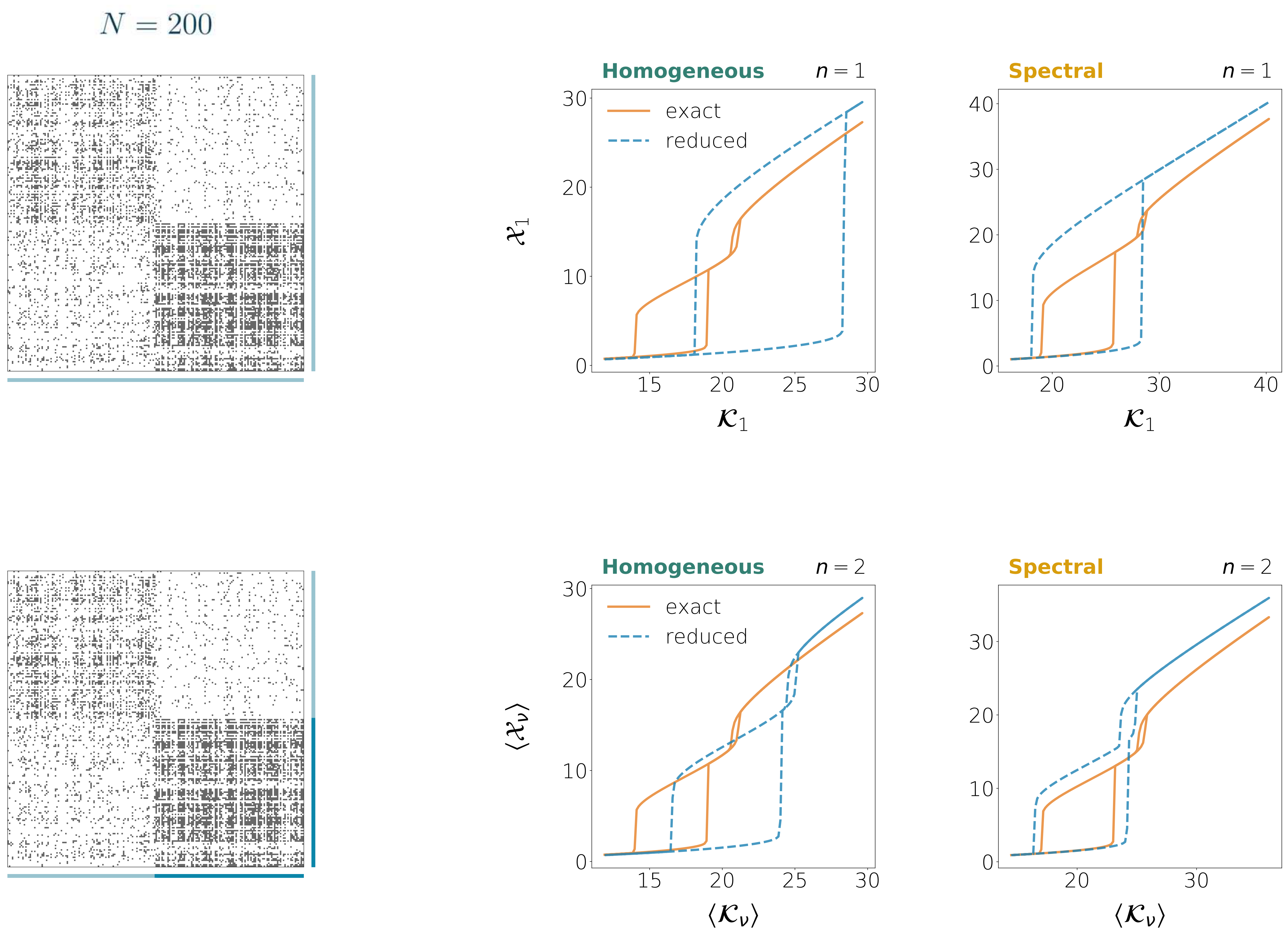
$N = 200$
Homogeneous
$n = 1$
Spectral
$n = 1$
exact
reduced
$\mathcal{X}_1$
$\mathcal{K}_1$
$\mathcal{K}_1$
Homogeneous
$n = 2$
Spectral
$n = 2$
exact
reduced
$\langle \mathcal{X}_\nu \rangle$
$\langle \mathcal{K}_\nu \rangle$
$\langle \mathcal{K}_\nu \rangle$

$N = 200$



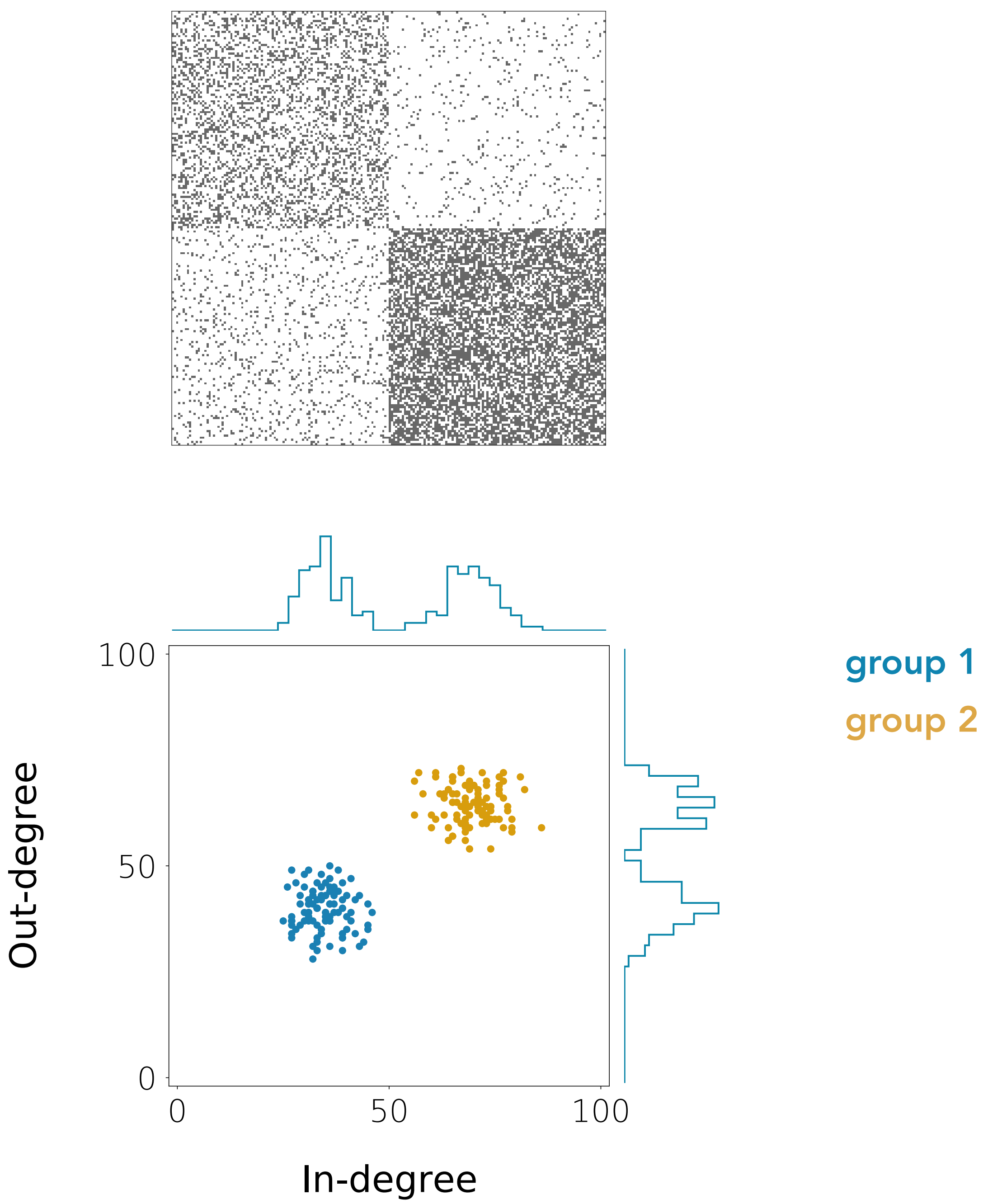


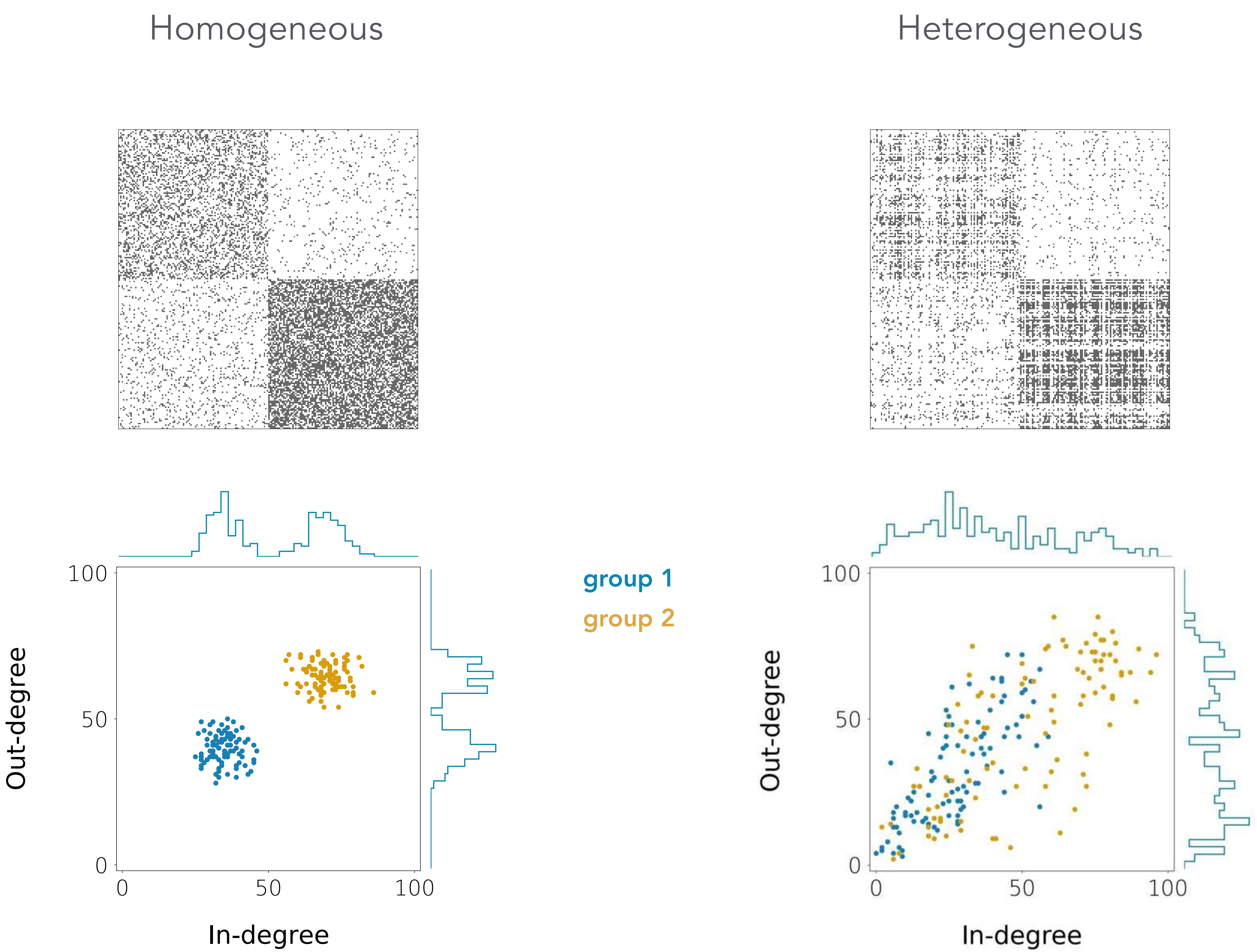
$N = 200$
Homogeneous
$n = 1$
exact
reduced
$\mathcal{X}_1$
$\mathcal{K}_1$
Spectral
$n = 1$
$\mathcal{K}_1$

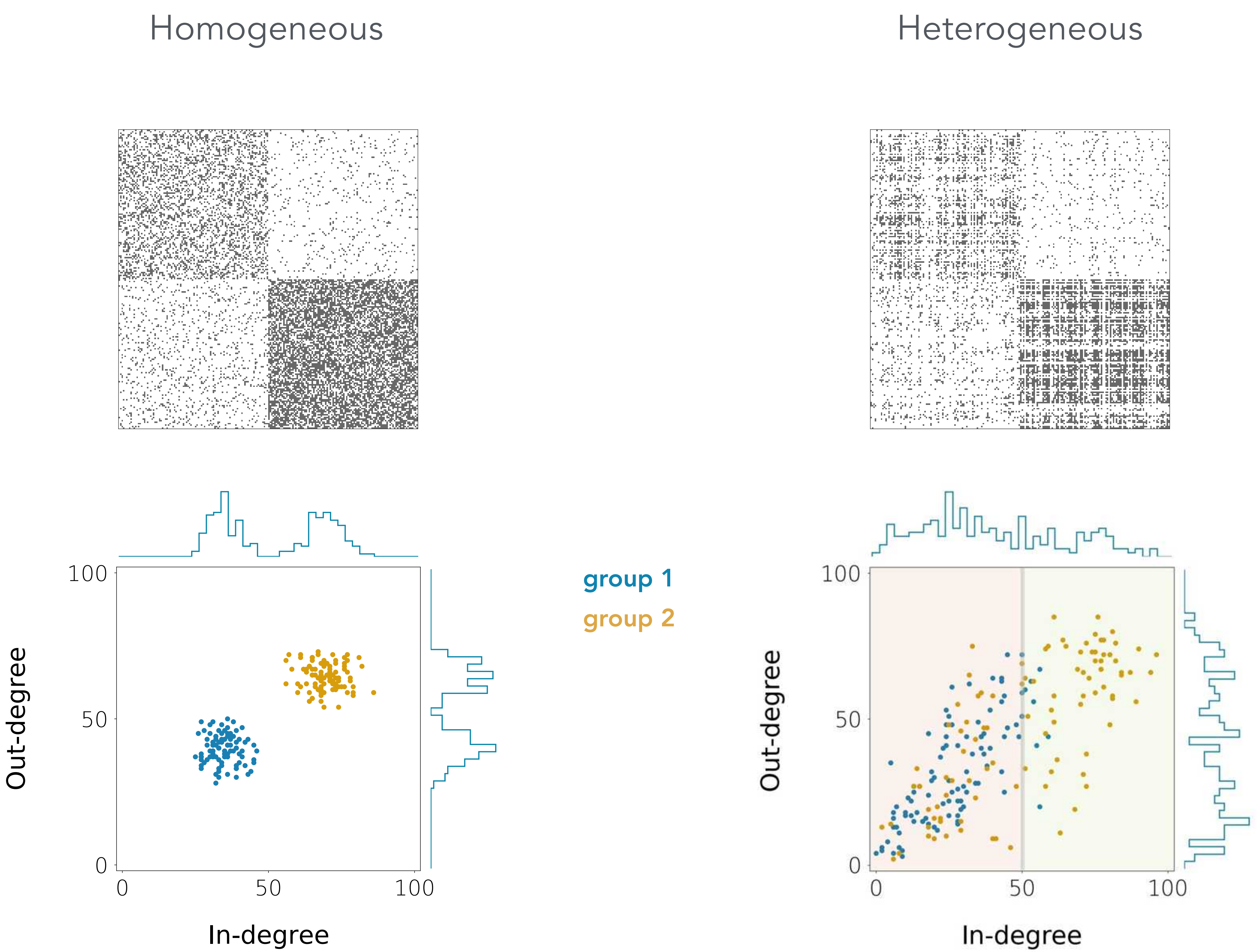


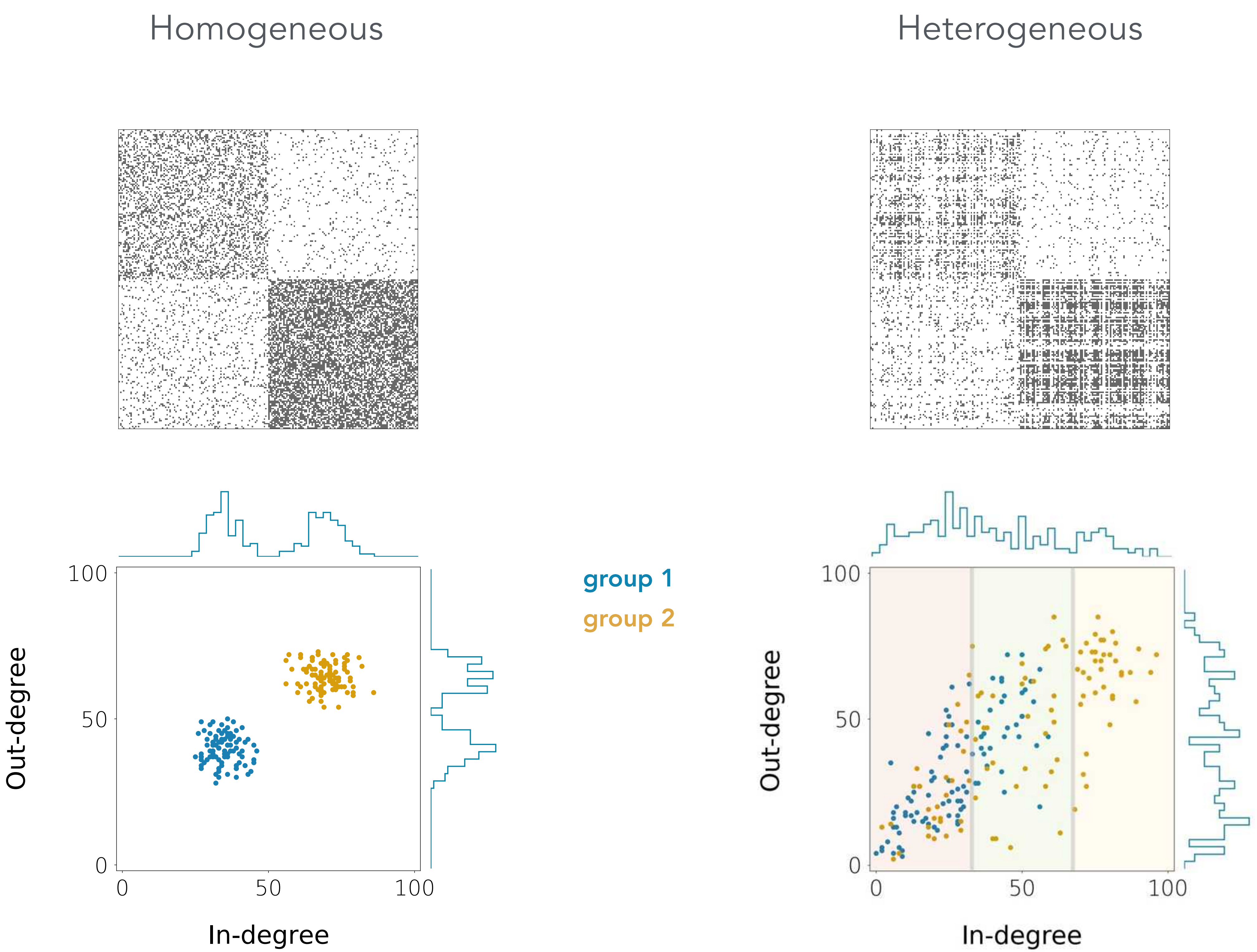
$N = 200$
Homogeneous
$n = 1$
exact
reduced
$\mathcal{X}_1$
$\mathcal{K}_1$
Spectral
$n = 1$
$\mathcal{K}_1$
Homogeneous
$n = 2$
exact
reduced
$\langle \mathcal{X}_\nu \rangle$
$\langle \mathcal{K}_\nu \rangle$
Spectral
$n = 2$
$\langle \mathcal{K}_\nu \rangle$

Homogeneous

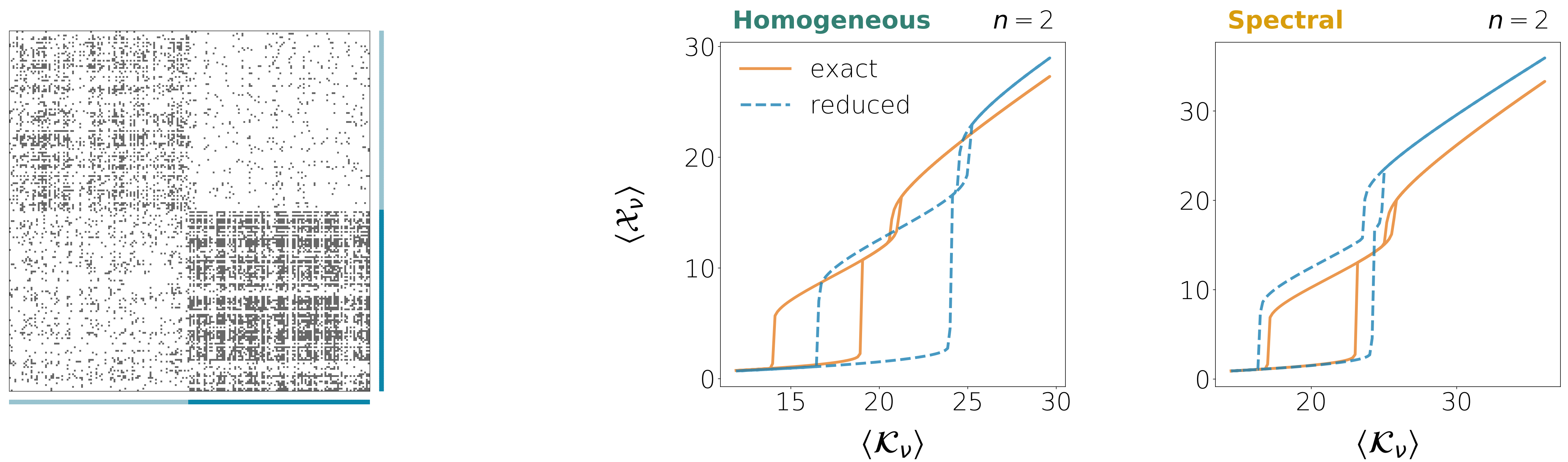



Homogeneous
Heterogeneous
100
50
0
0
50
100
Out-degree
In-degree
group 1
group 2
100
50
0
0
50
100
Out-degree
In-degree

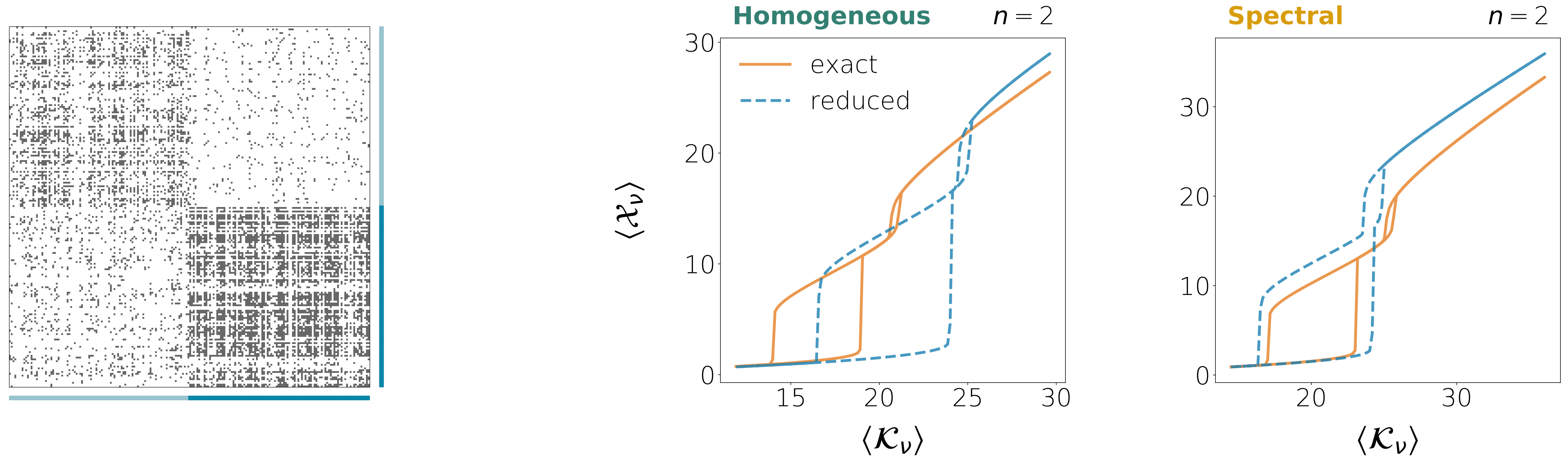

Homogeneous
Heterogeneous
group 1
group 2
Out-degree
In-degree
0
50
100
Out-degree
In-degree
0
50
100

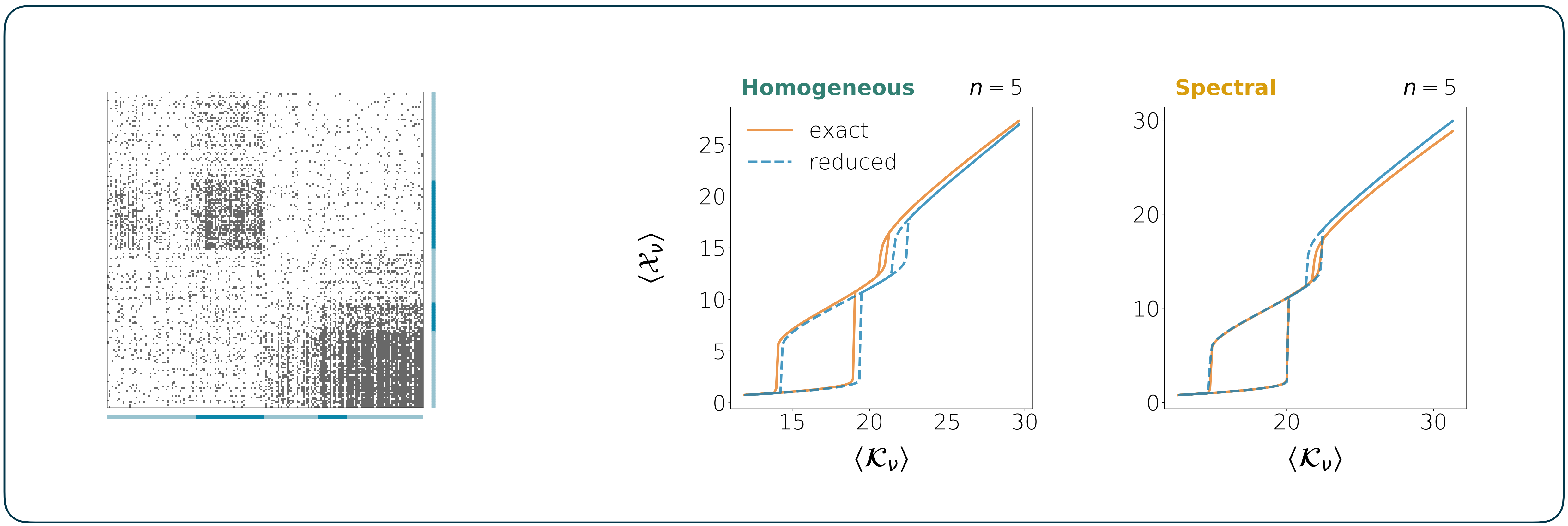


We can define more groups by partitioning the nodes within each group according to their connectivity properties

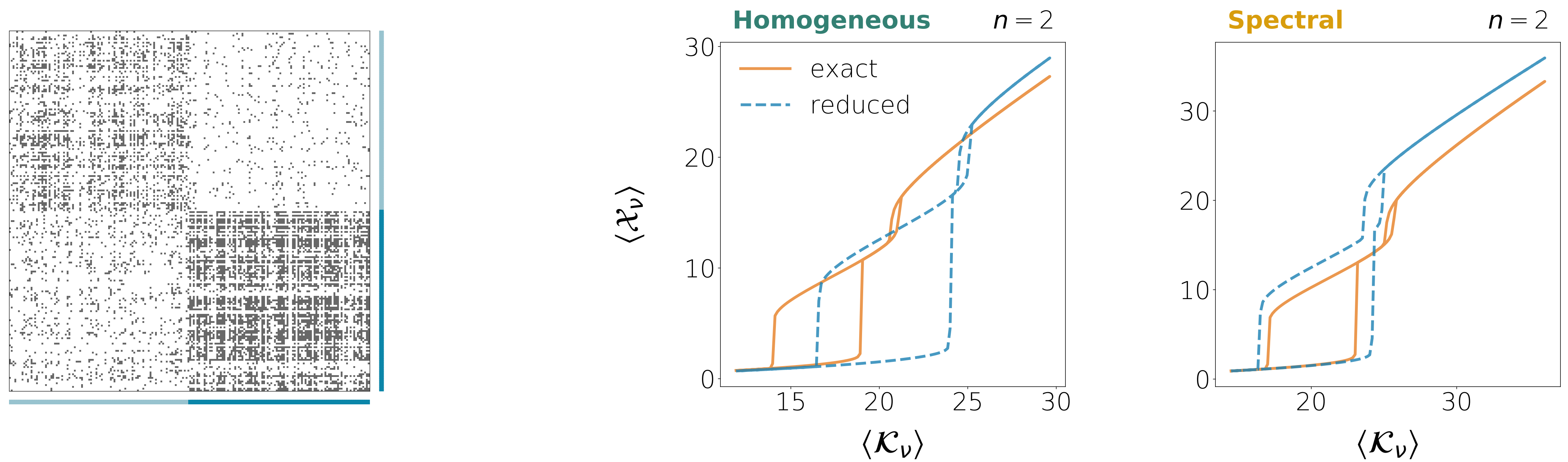


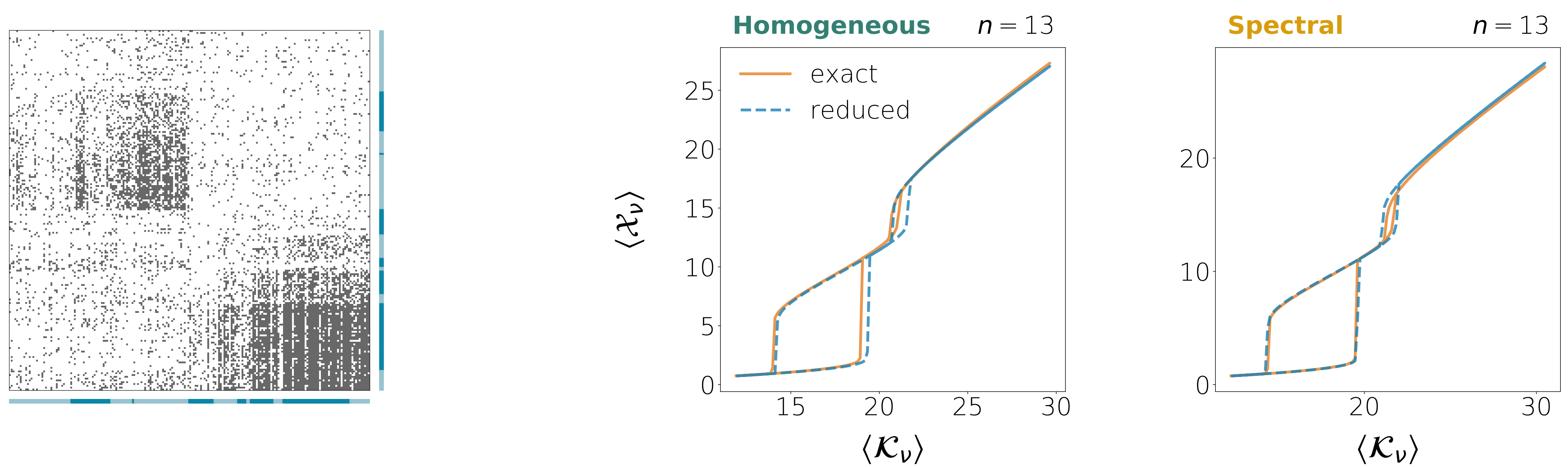
Homogeneous
n = 2
exact
reduced
⟨𝒳ν⟩
⟨𝒦ν⟩
Spectral
n = 2
⟨𝒦ν⟩





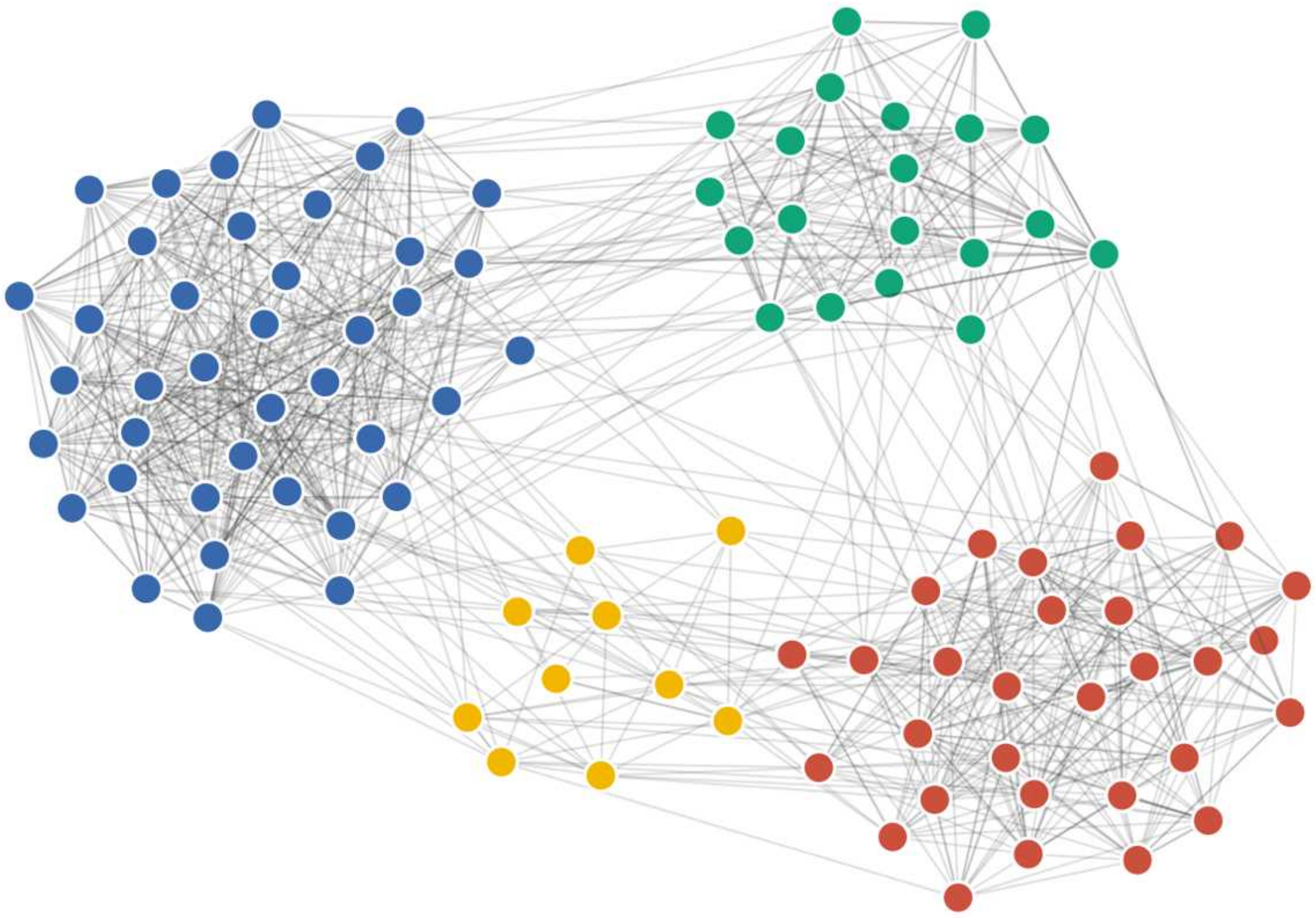
Partition refinement



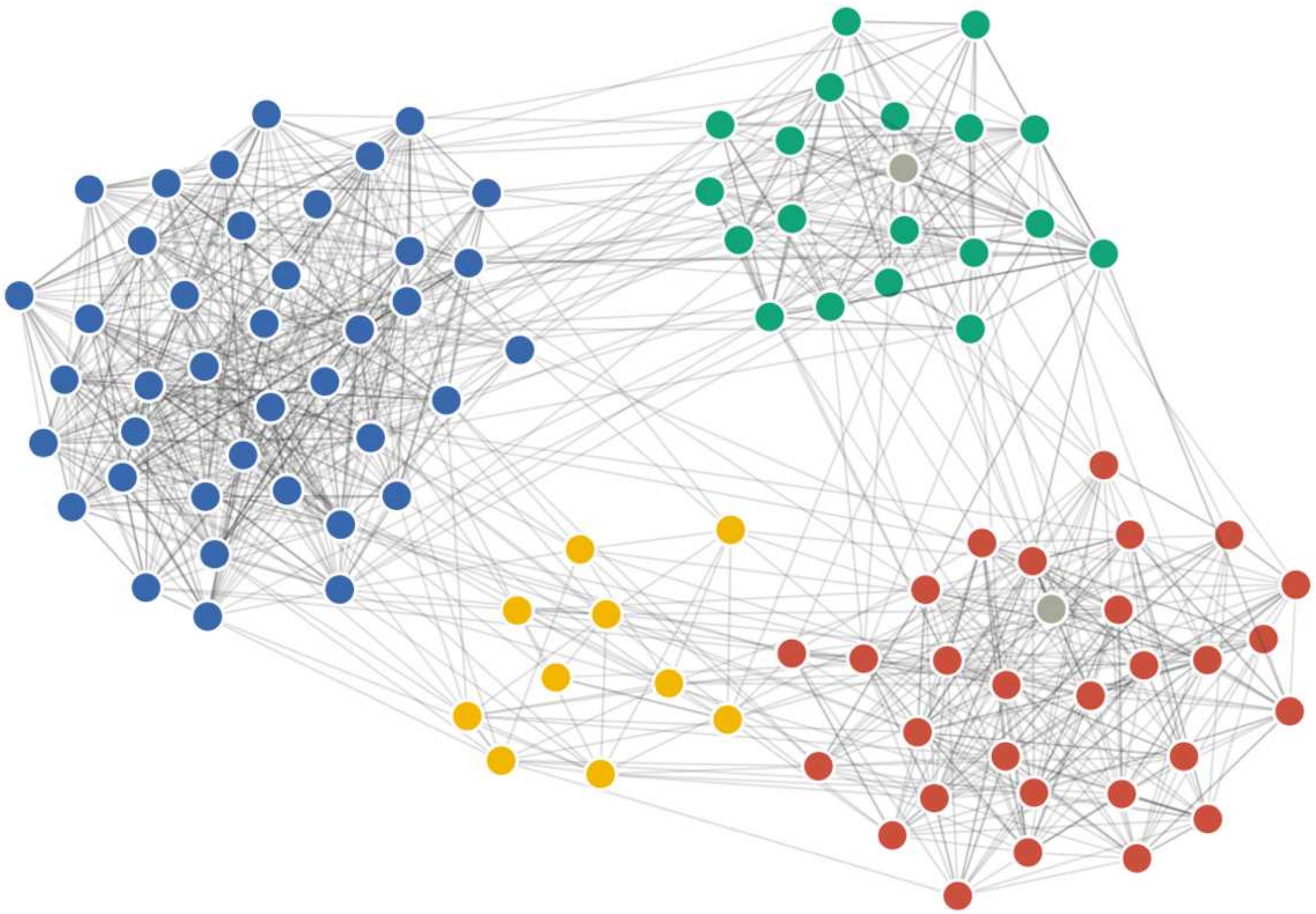


Partition refinement

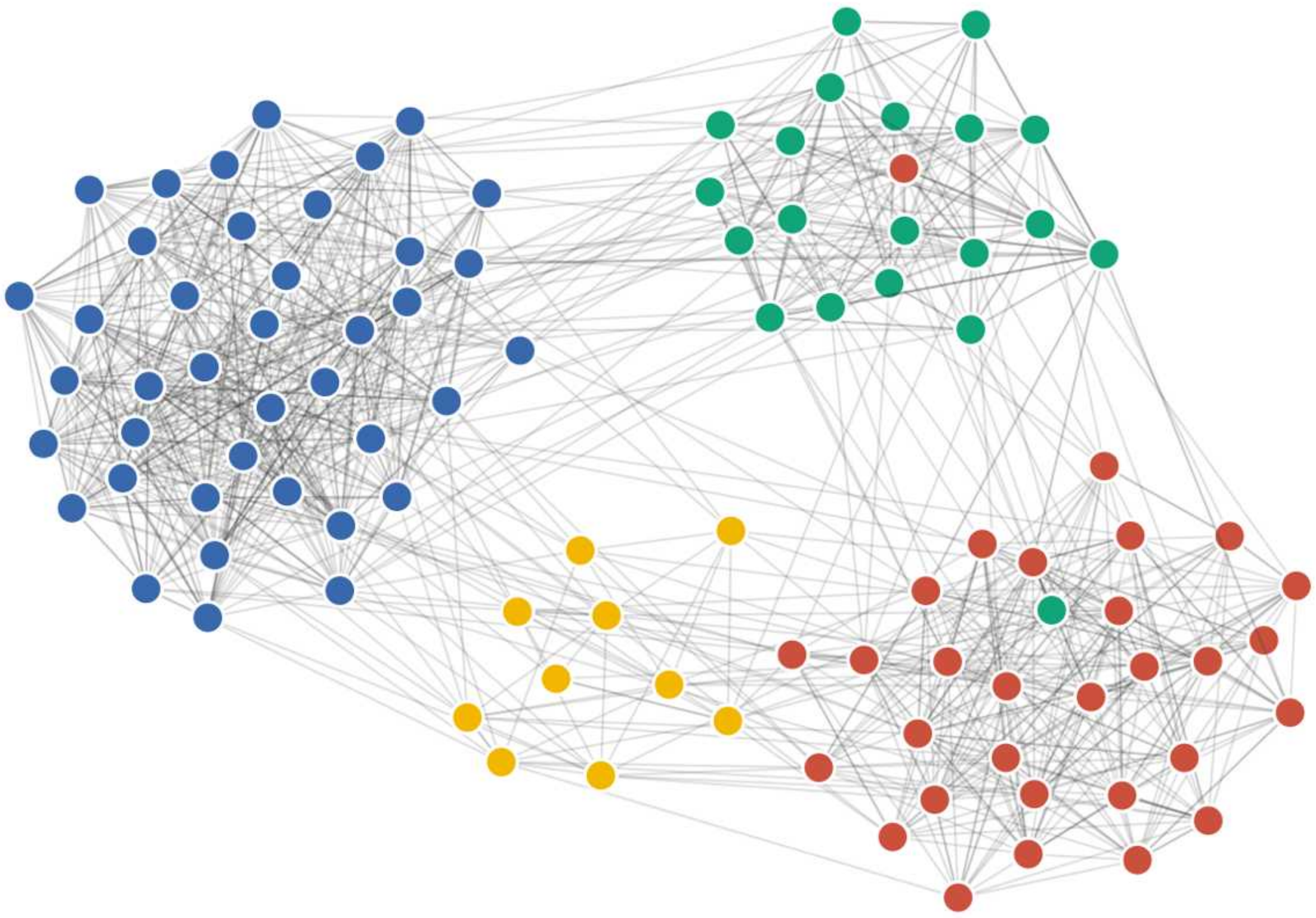
Sensitivity to partition choice



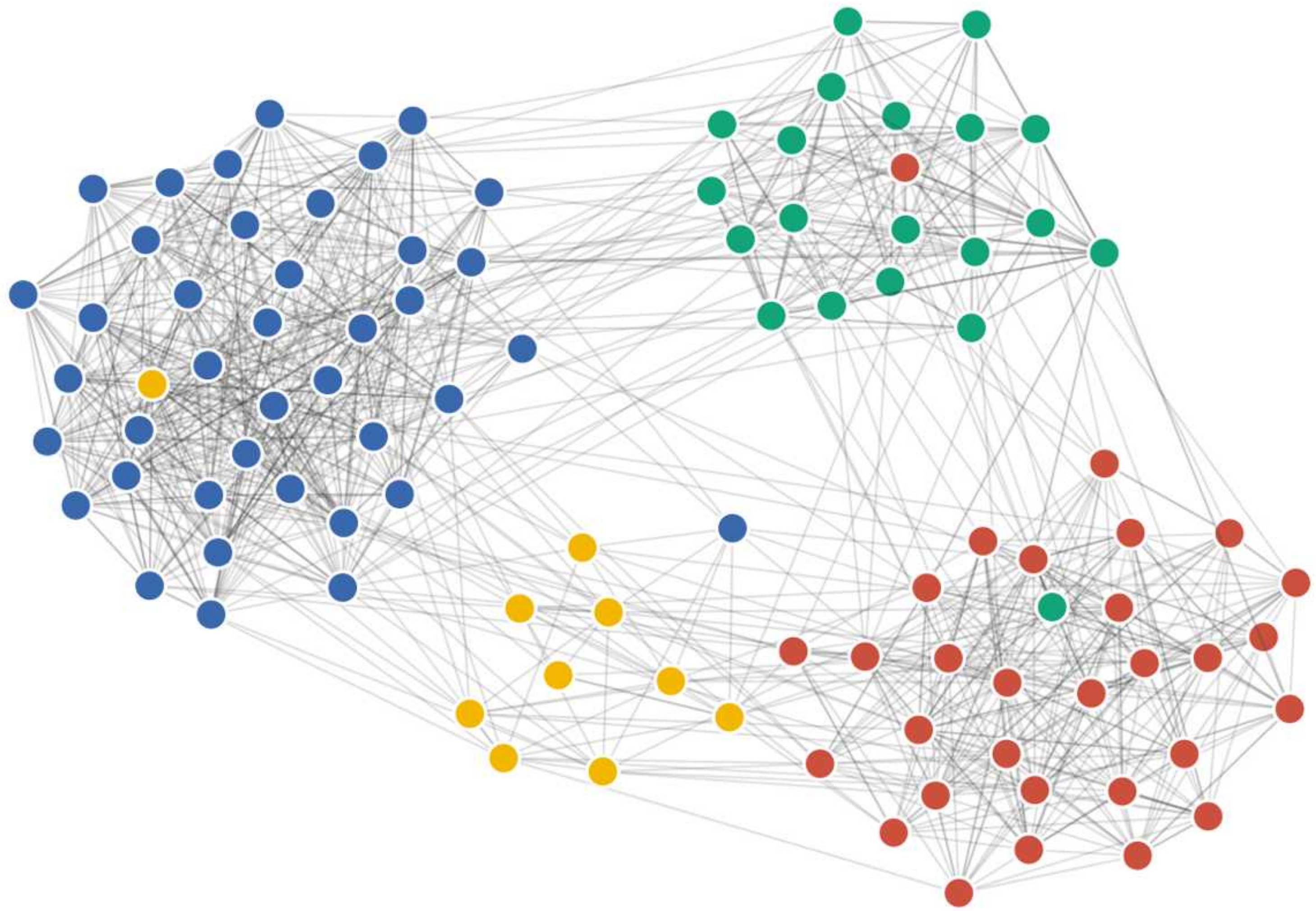
Sensitivity to partition choice



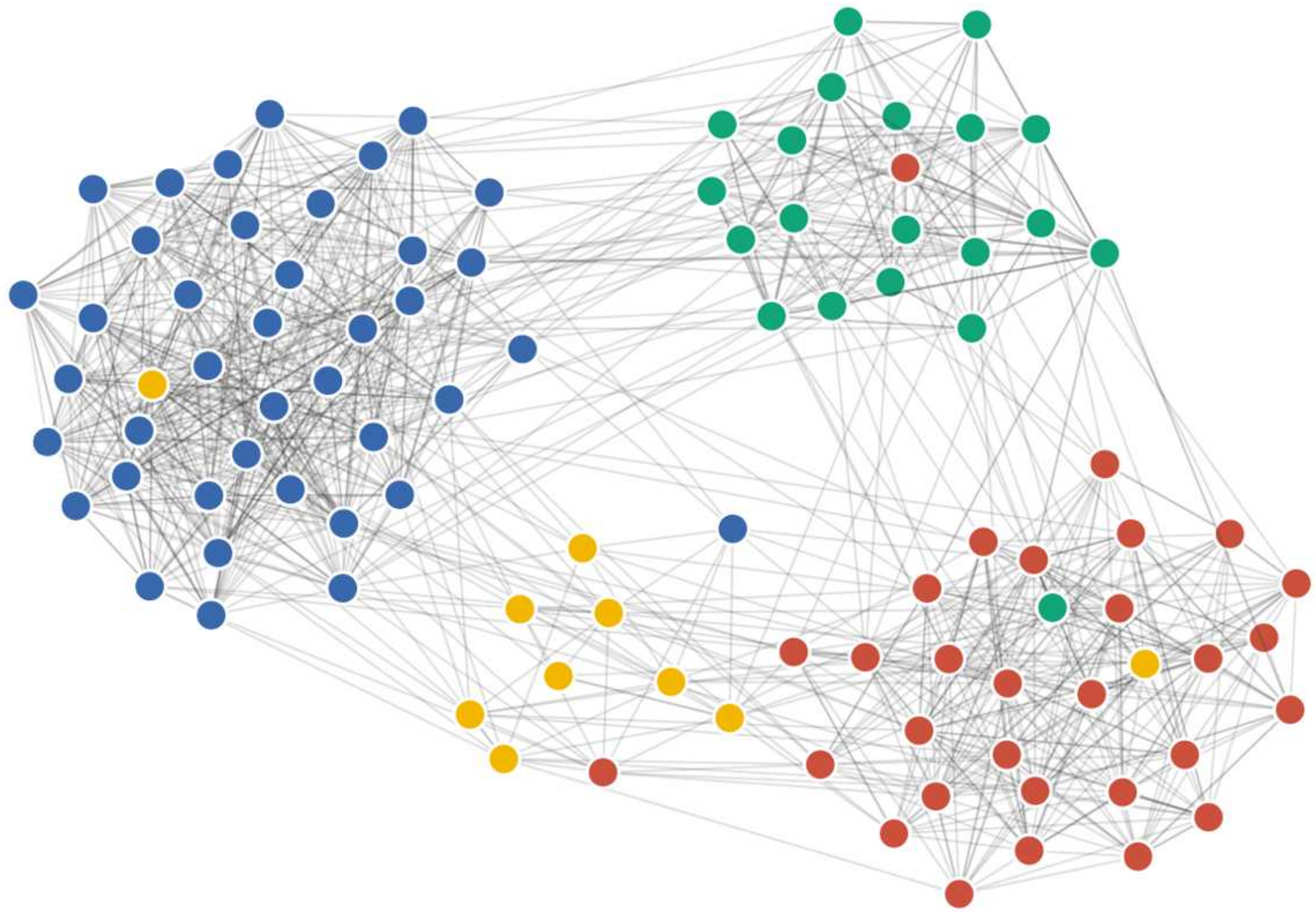
Sensitivity to partition choice



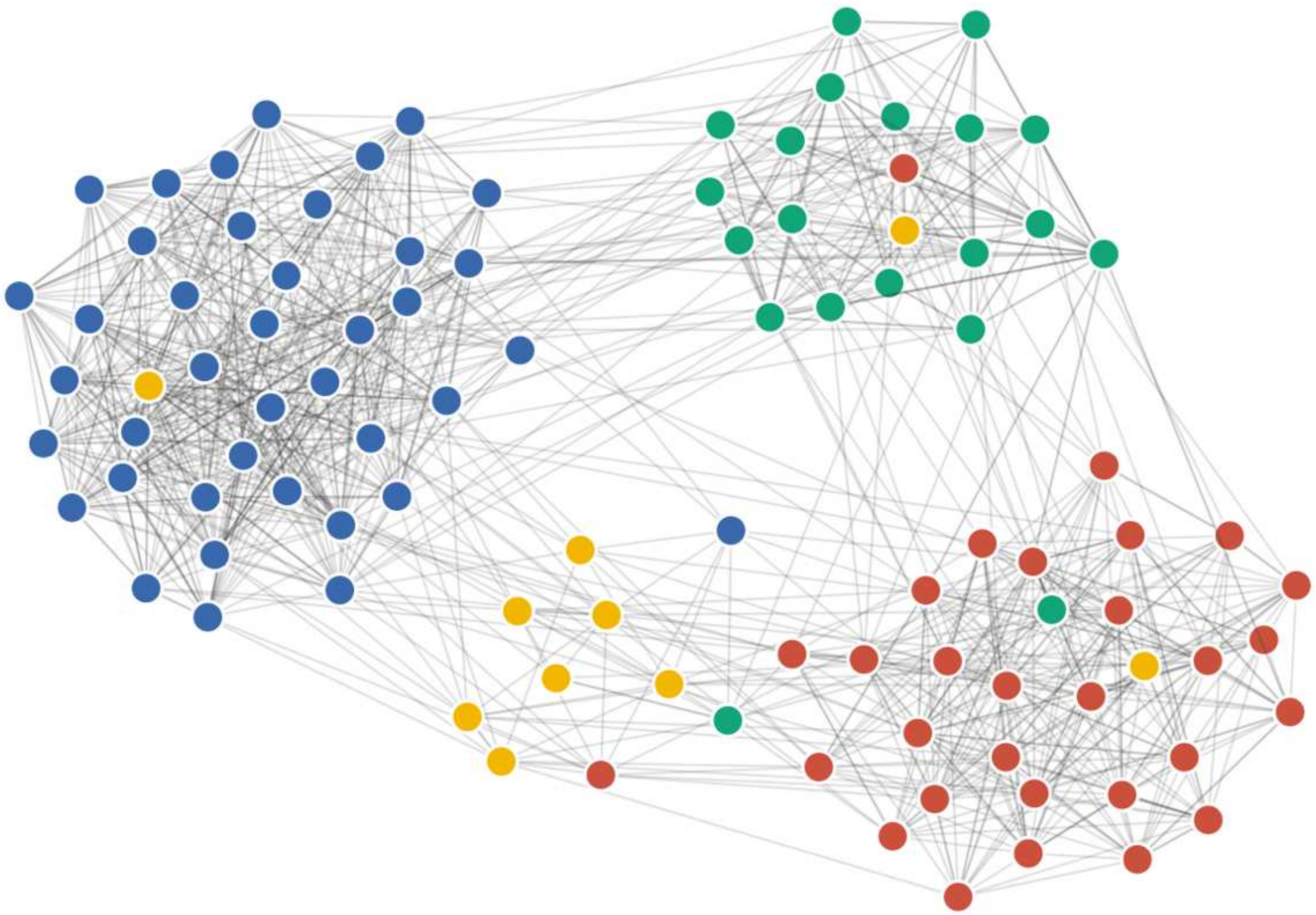
Sensitivity to partition choice



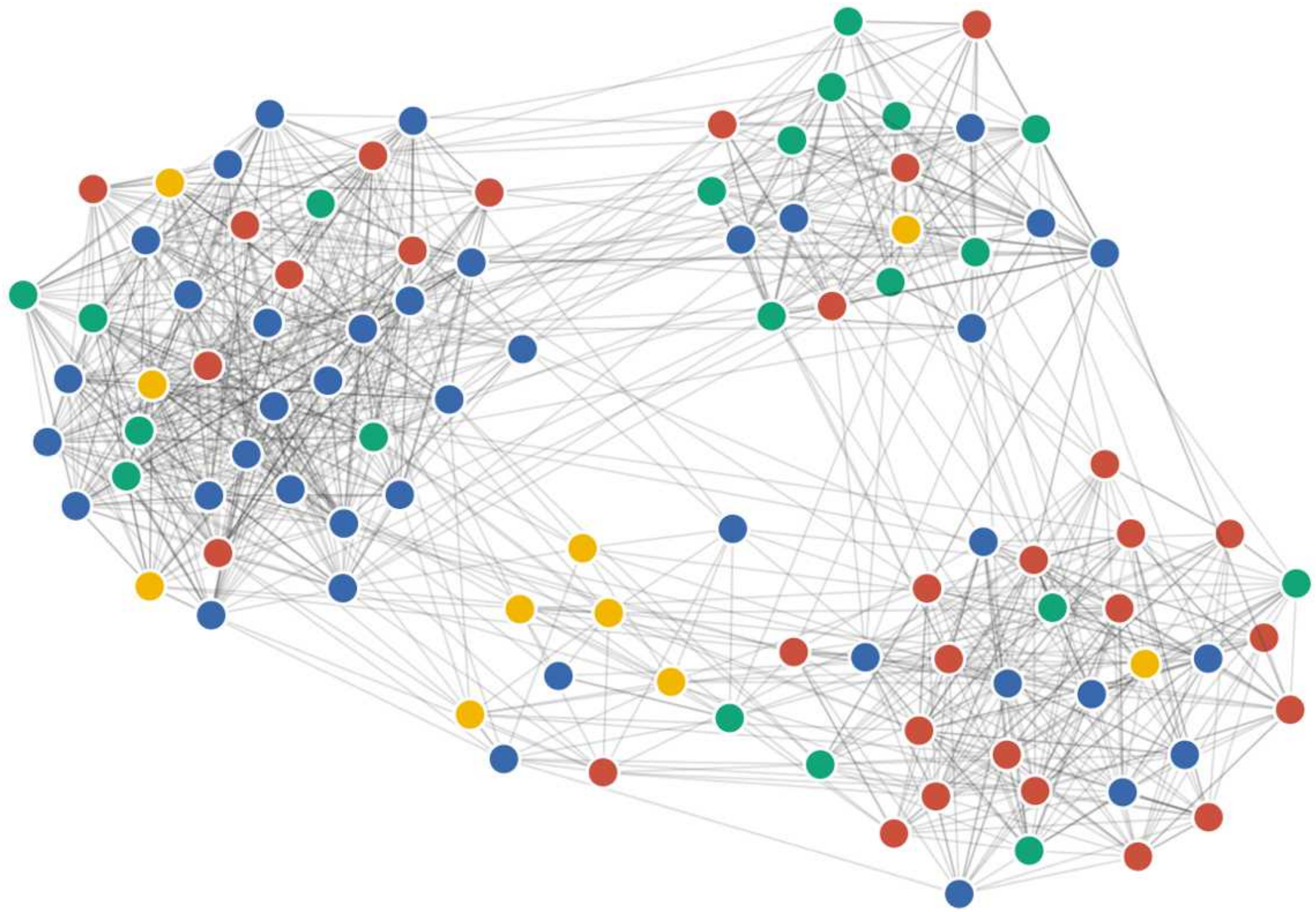
Sensitivity to partition choice



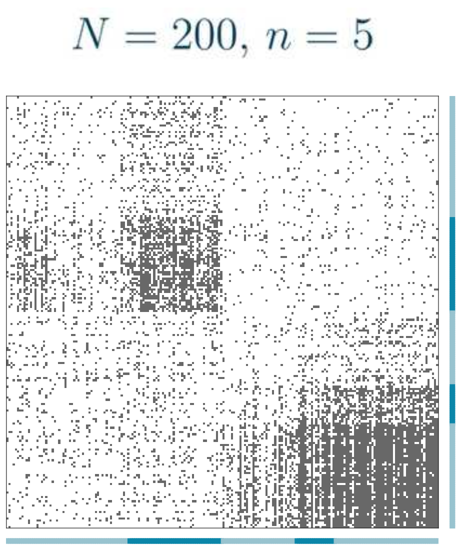
Sensitivity to partition choice

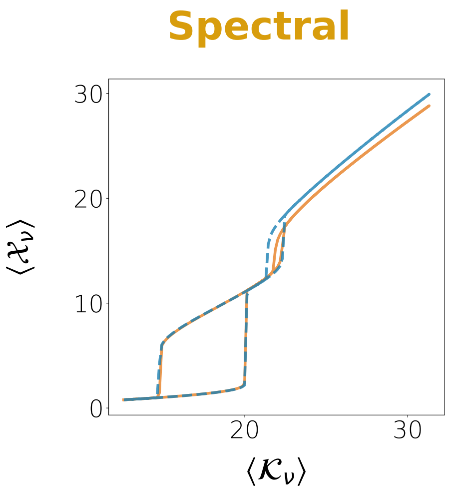


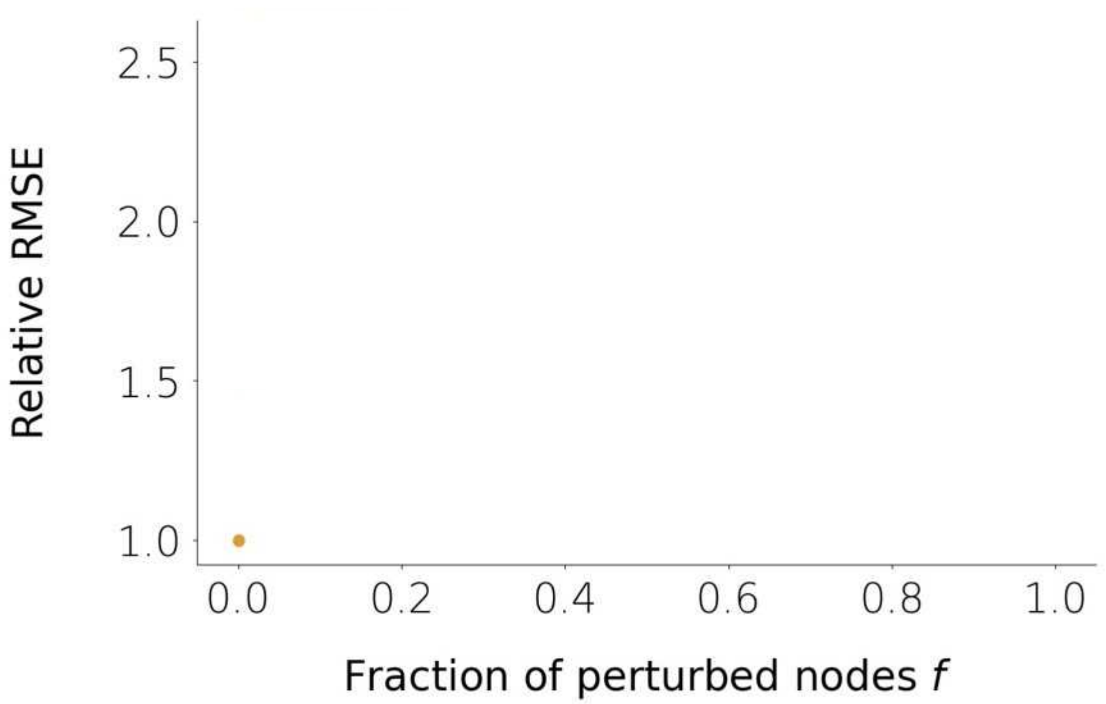
Sensitivity to partition choice



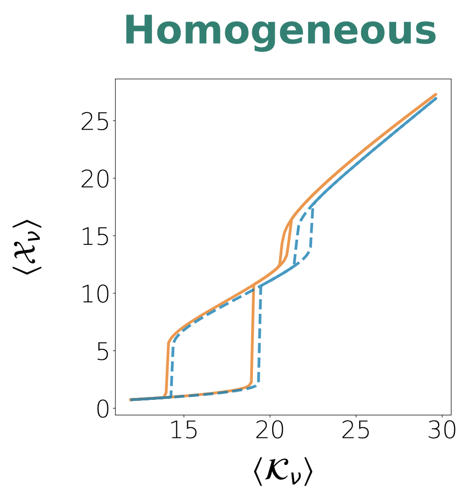
Sensitivity to partition choice

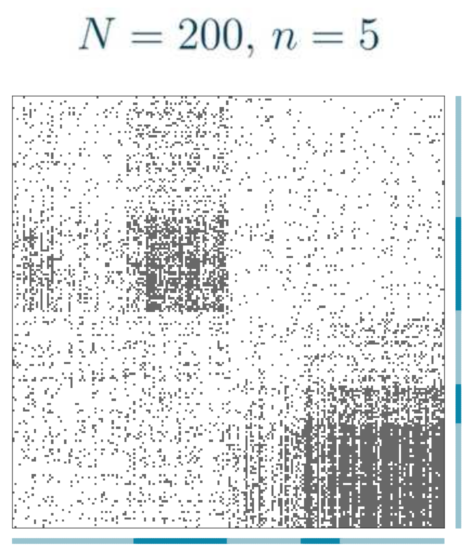


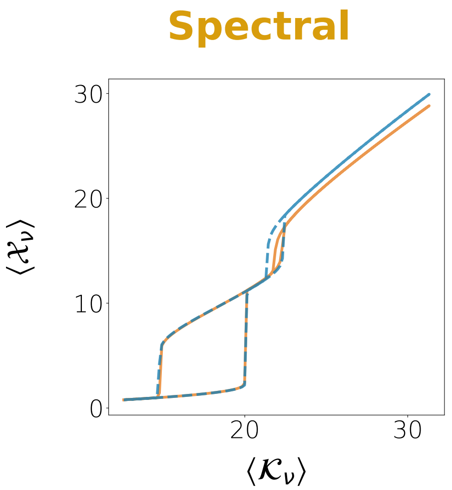


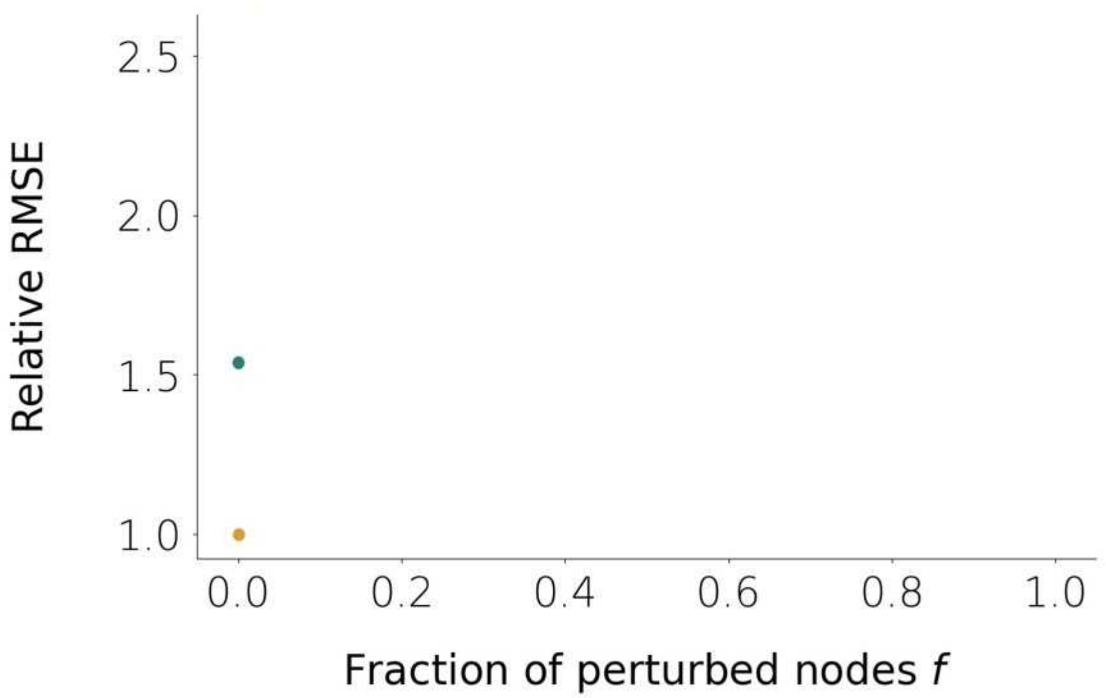


Sensitivity to partition choice

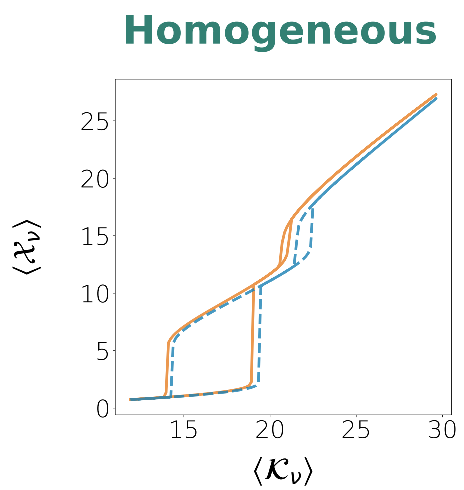


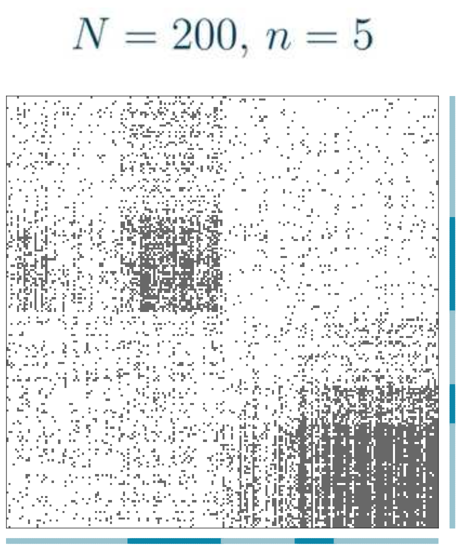


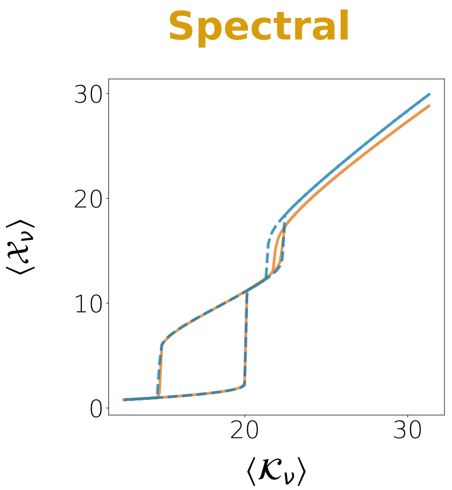


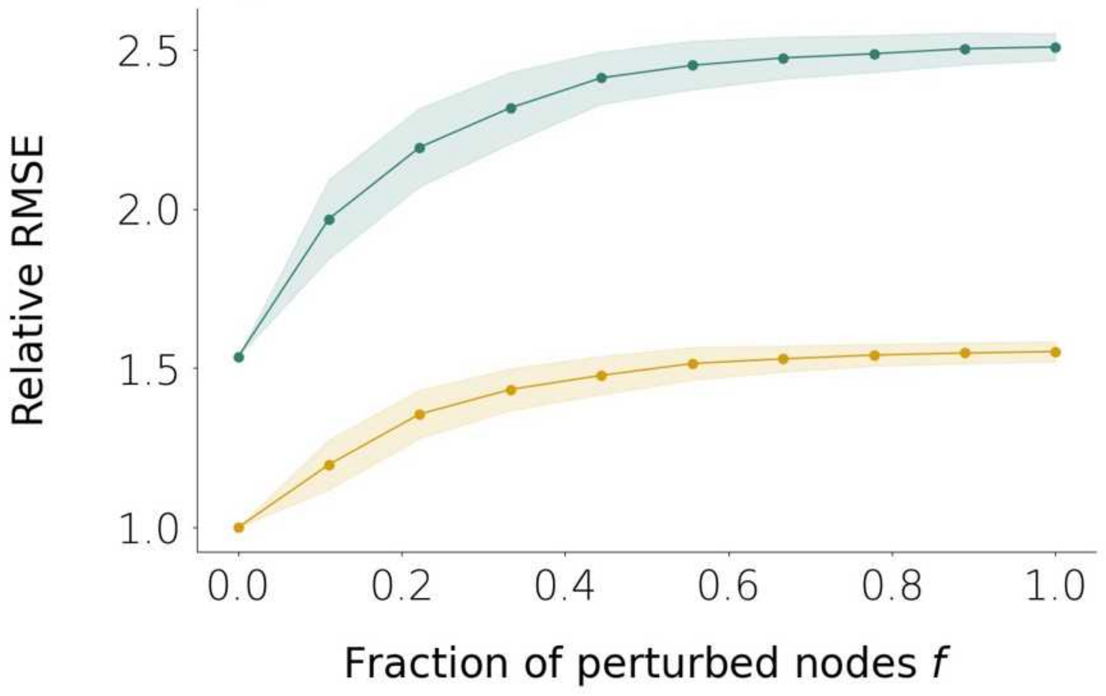


Sensitivity to partition choice









To summarize…

- Dimension reduction can be used to extract dynamical properties of complex networks such as bifurcation points
- The Spectral reduction
  - can be applied to directed interaction matrices
  - performs well on heterogeneous networks
  - is robust to perturbations of node grouping