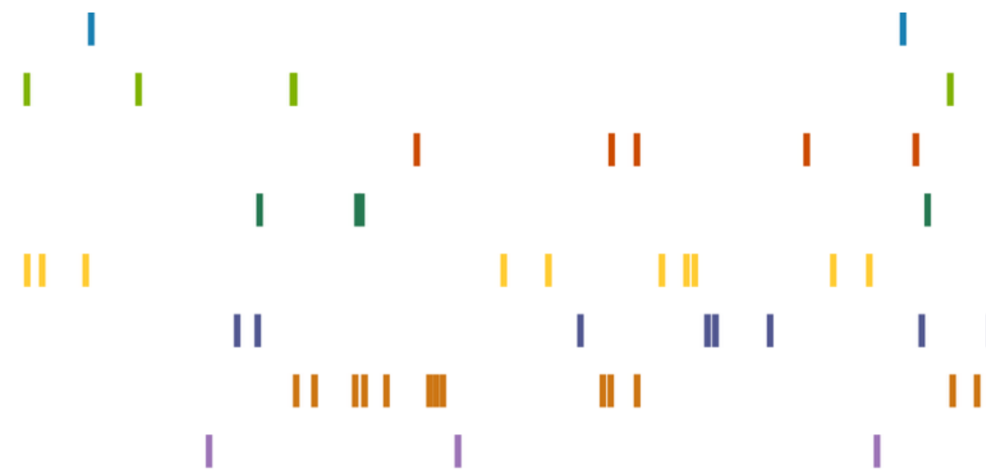
Firing rate distributions in plastic networks of spiking neurons



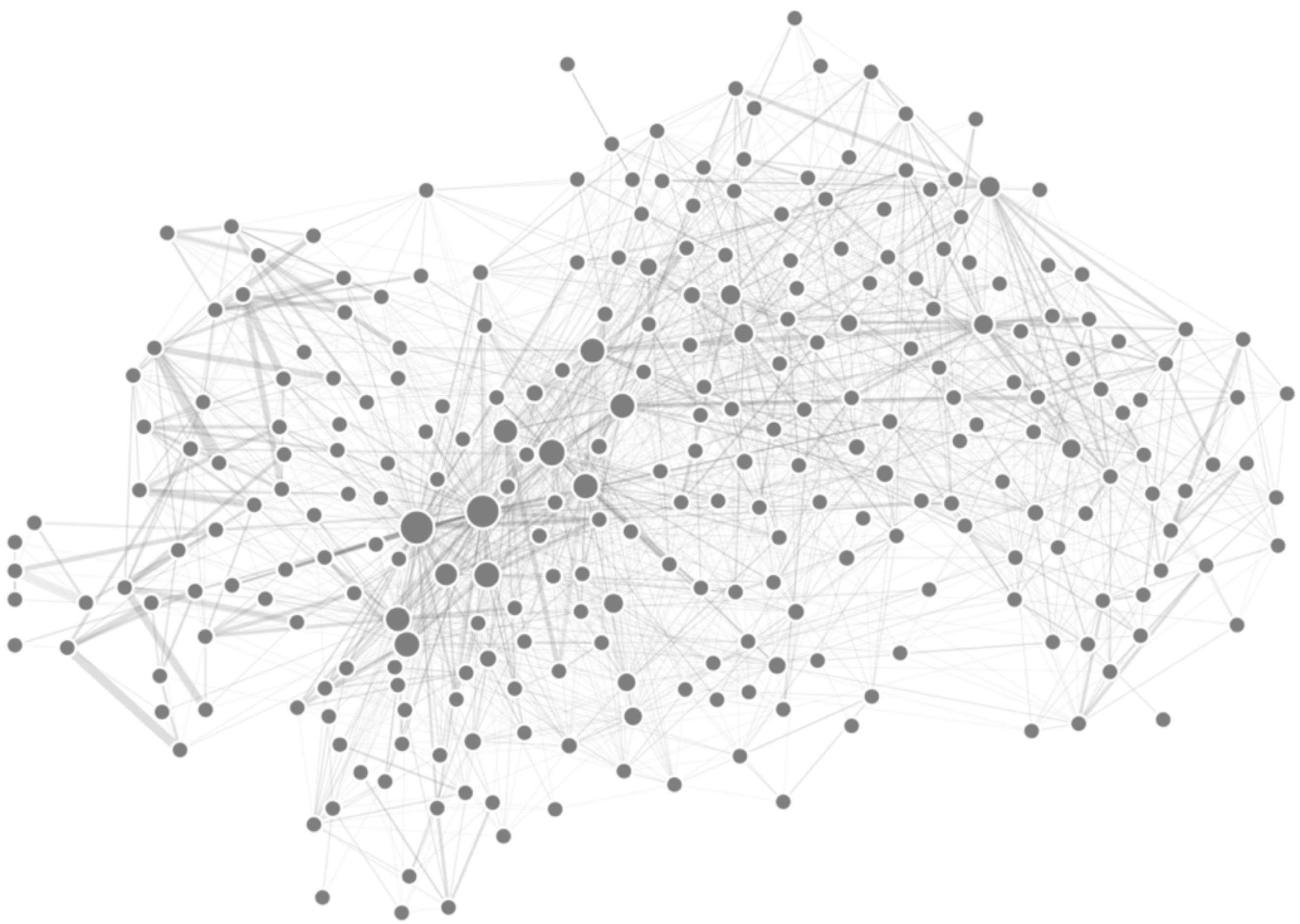
Marina Vegué

Antoine Allard

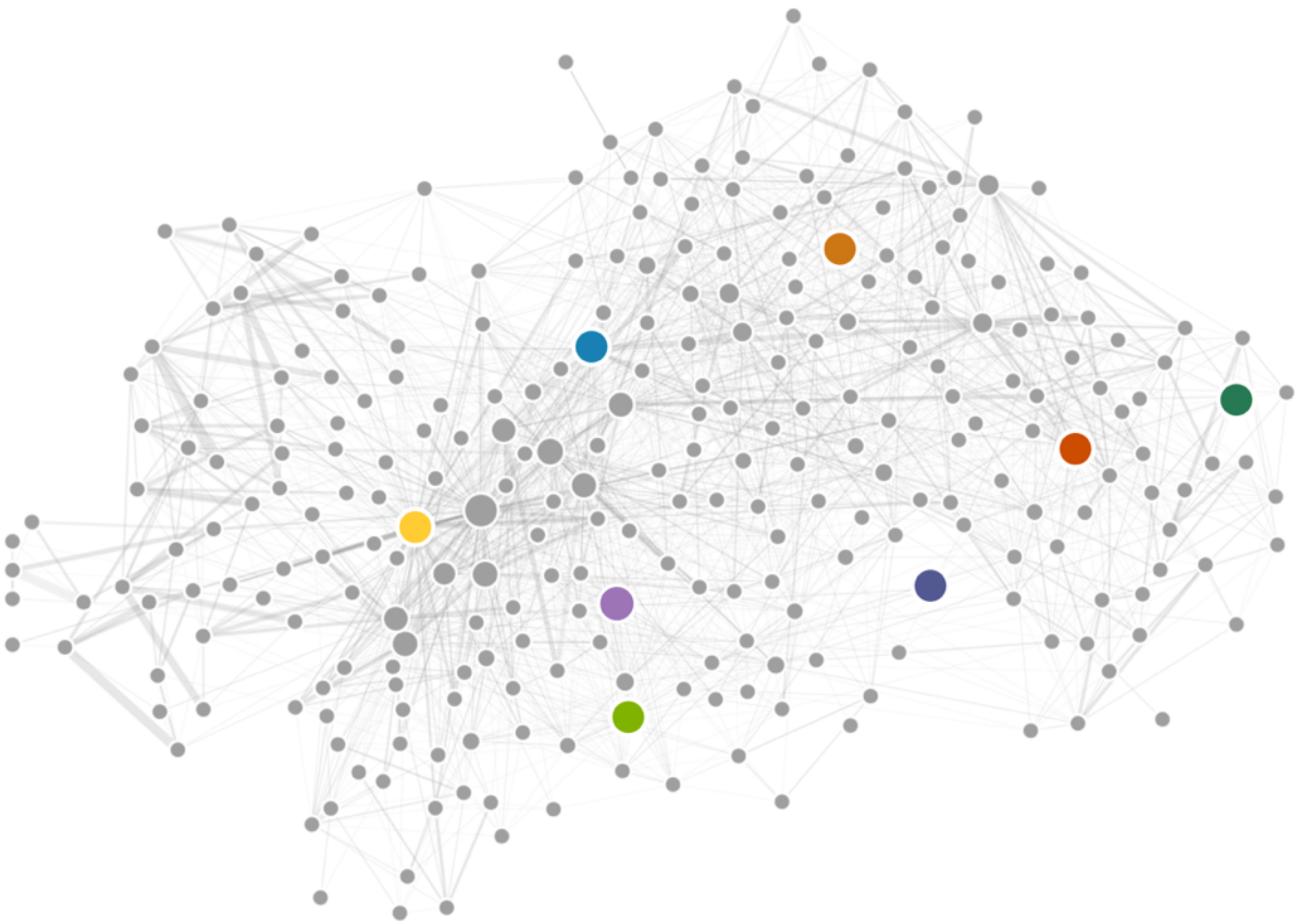
Patrick Desrosiers

Dynamica Research Group

Université Laval, Québec, Canada



Neuronal network



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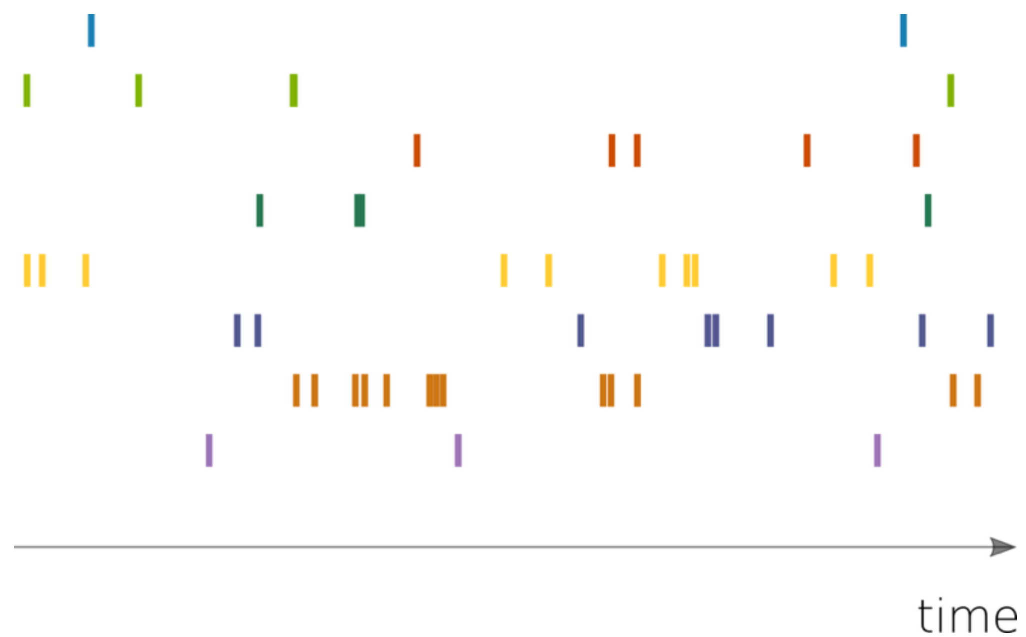
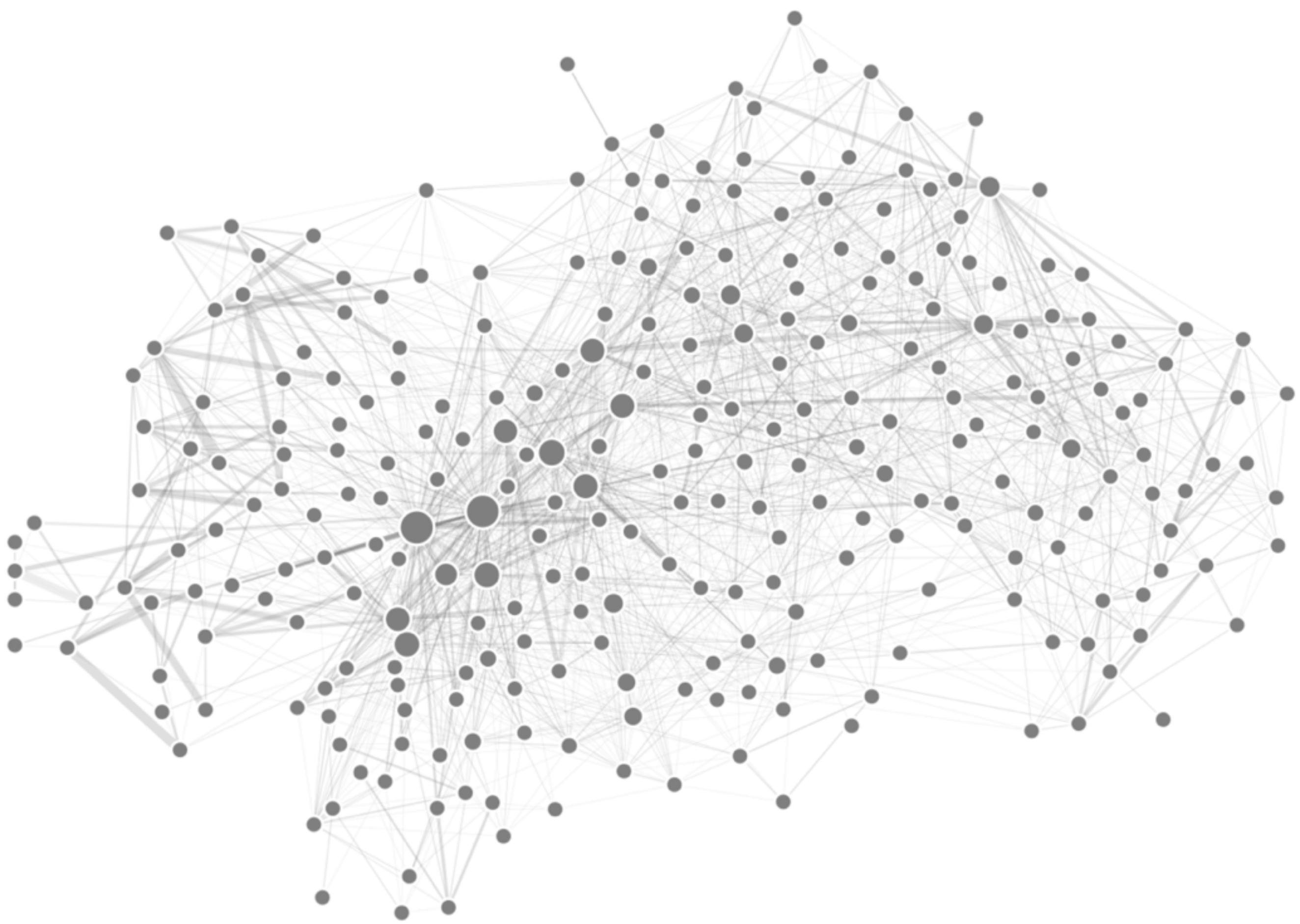
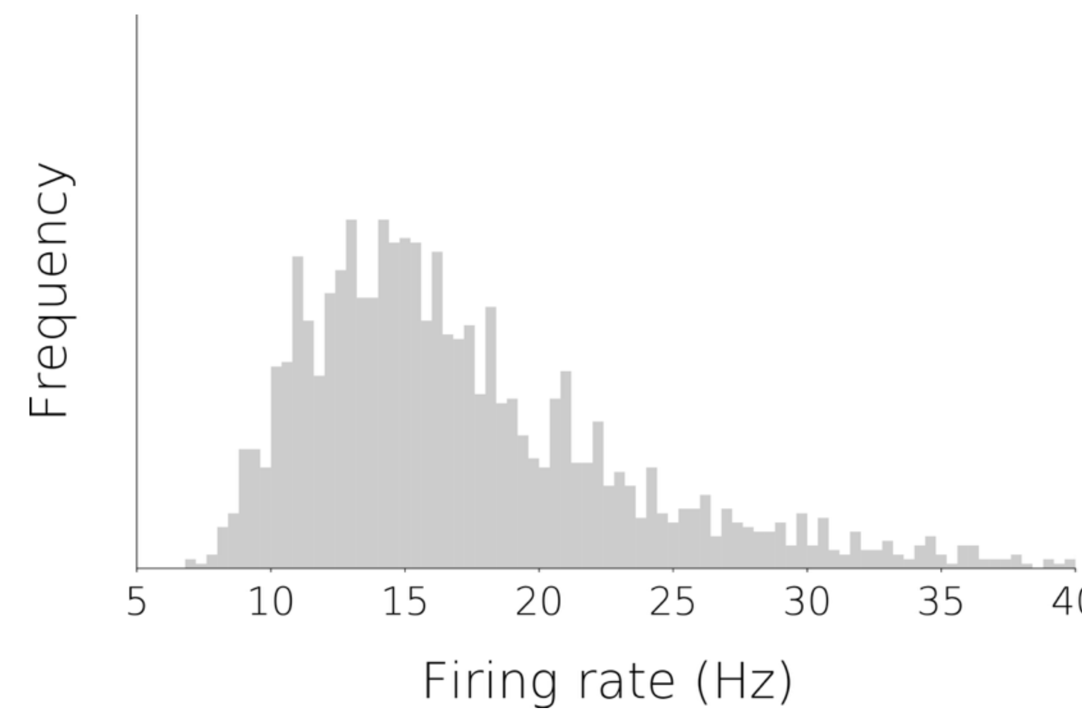


Neuronal activity





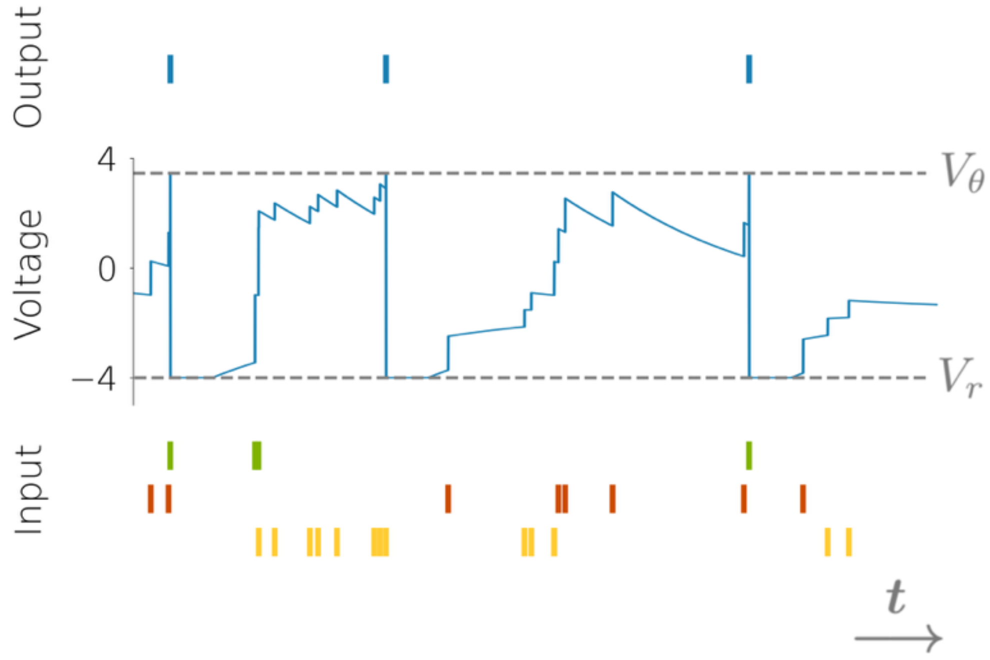
Network structure ↔ Activity distribution

Neuronal dynamics: leaky integrate-and-fire model

$$V_i'(t) = -V_i(t) + I_i(t)$$

$$I_i(t) = \sum_{j=1}^{K_i} w_{ij}(t) \sum_k \delta(t - t_j^k)$$

V_i	voltage	I_i	synaptic input
K_i	in-degree	w_{ij}	synaptic weight
V_θ	spike threshold	V_r	reset potential



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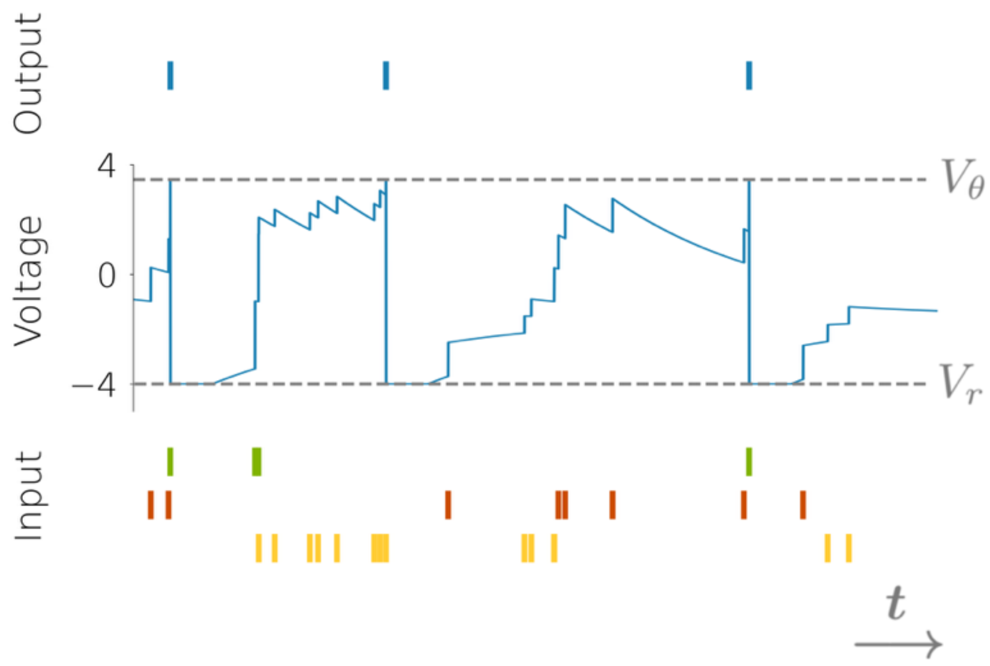

This problem has been studied* for networks with

a fixed in/out-degree distribution

and

homogeneous and constant weights: $w_{ij}(t) = w$ for all i, j, t

* N. Brunel. *J Comput Neurosci*, 8(3): 183-208, 2000
A. Roxin et al. *J Neurosci*, 31(45): 16217-16226, 2011
M. Vegué and A. Roxin. *Phys Rev E*, 100(2): 022208, 2019

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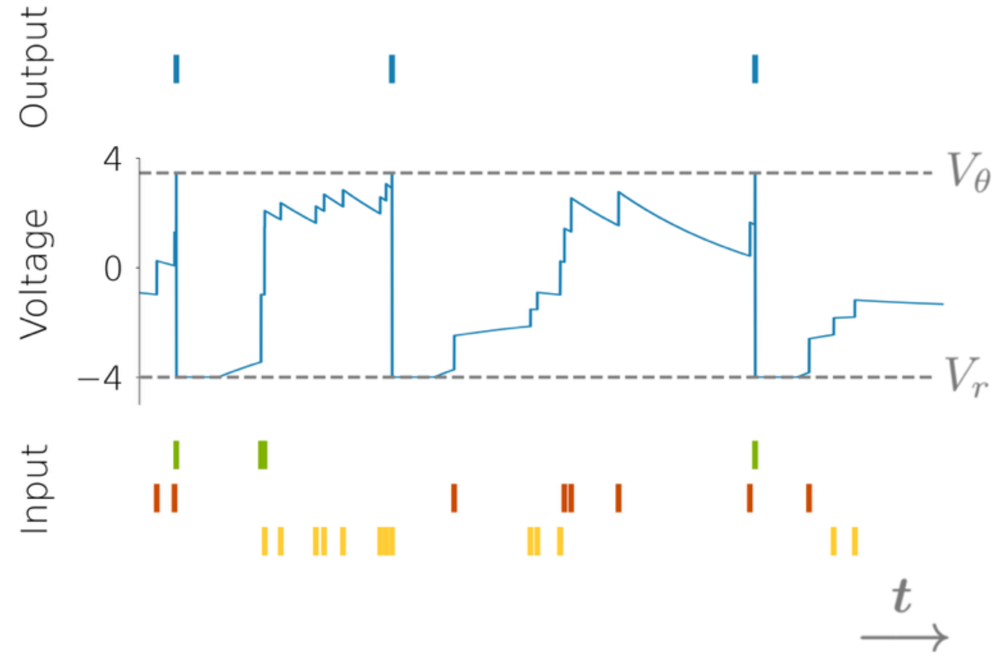

Synaptic weights

Binary scaffold

Neuronal dynamics: leaky integrate-and-fire model

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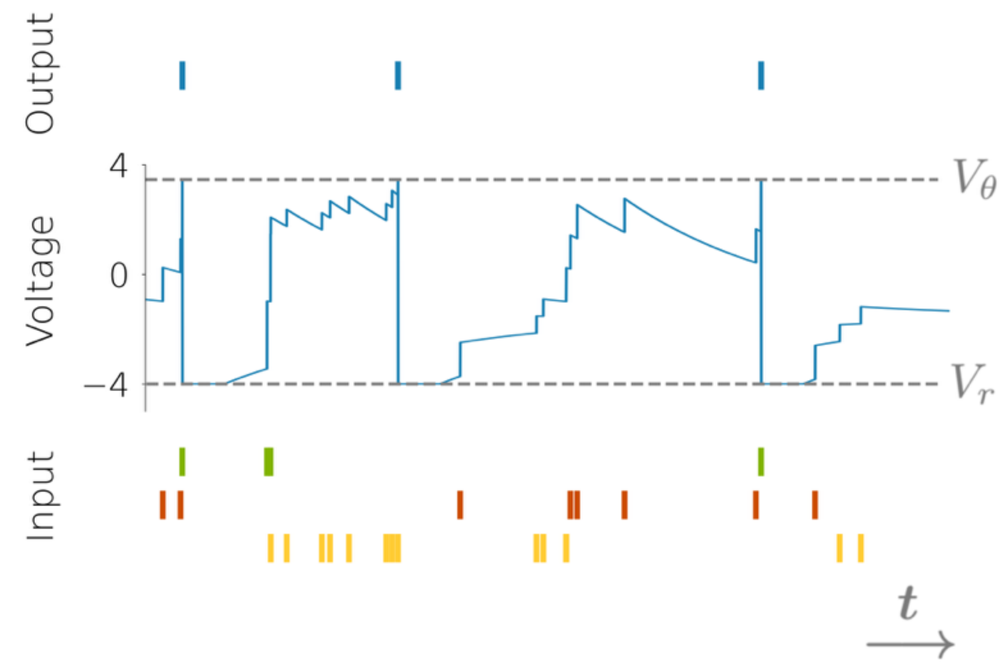
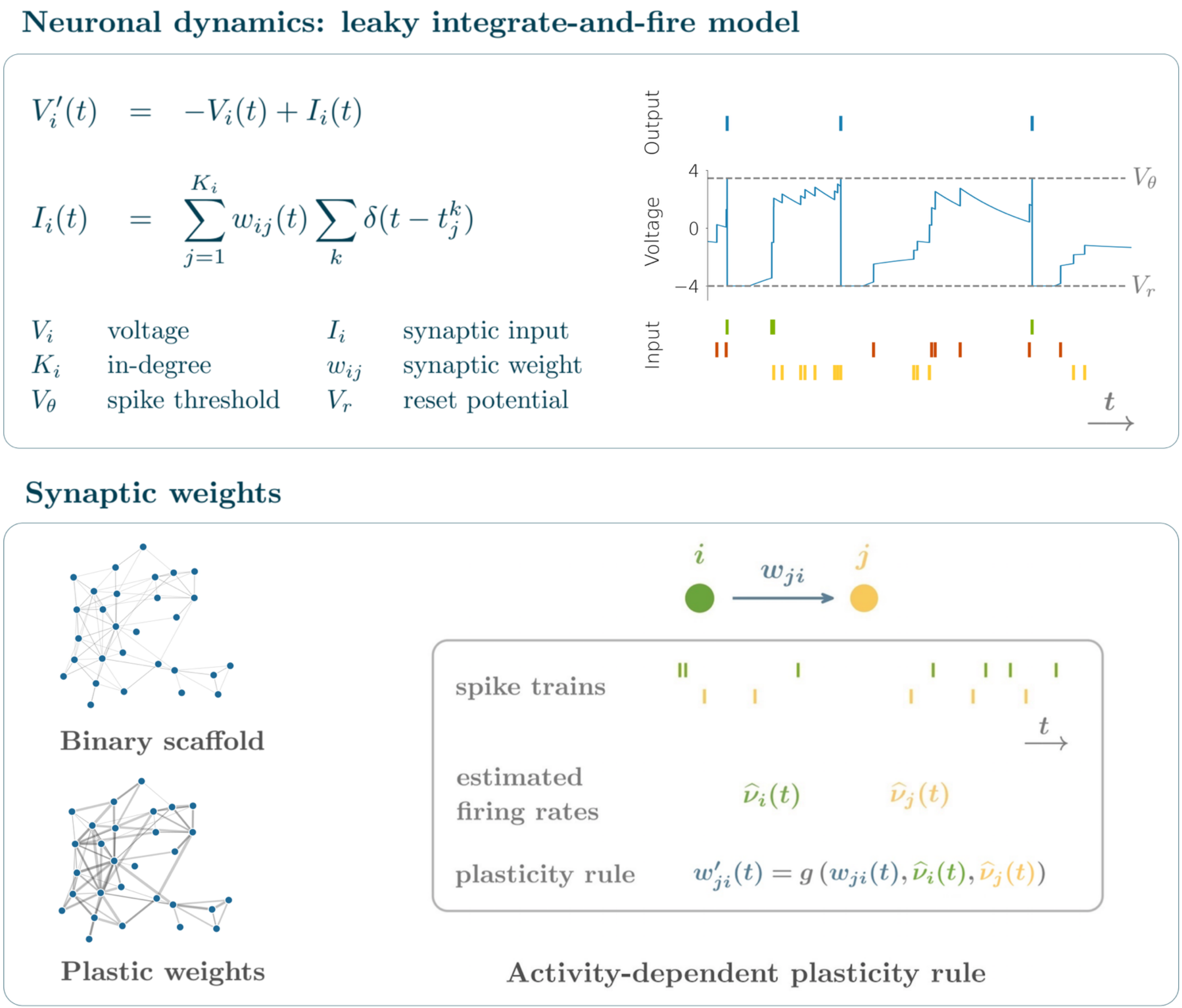


Synaptic weights

Binary scaffold

Plastic weights



Neuronal dynamics: leaky integrate-and-fire model
$V_i'(t) = -V_i(t) + I_i(t)$
$I_i(t) = \sum_{j=1}^{K_i} w_{ij}(t) \sum_k \delta(t - t_j^k)$
V_i voltage
K_i in-degree
V_θ spike threshold
I_i synaptic input
w_{ij} synaptic weight
V_r reset potential
Output
Voltage
Input
4
0
−4
V_θ
V_r
t
Synaptic weights
Binary scaffold
Plastic weights
i
w_{ji}
j
spike trains
t
estimated firing rates
$\widehat{\nu}_i(t)$
$\widehat{\nu}_j(t)$
plasticity rule
$w_{ji}'(t) = g\left(w_{ji}(t), \widehat{\nu}_i(t), \widehat{\nu}_j(t)\right)$
Activity-dependent plasticity rule

Neuronal dynamics: leaky integrate-and-fire model

$$V_i'(t) = -V_i(t) + I_i(t)$$

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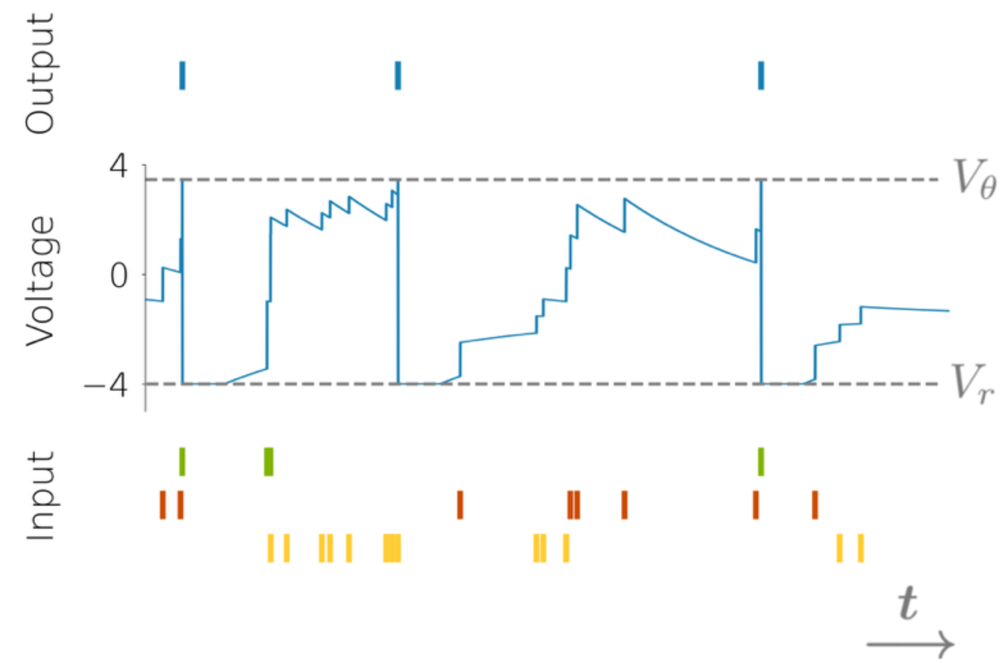
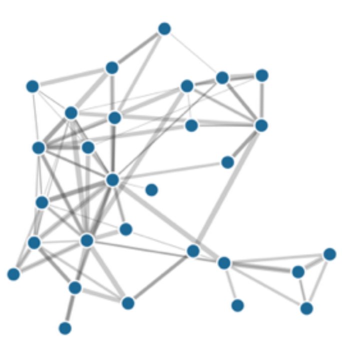


Synaptic weights

Binary scaffold



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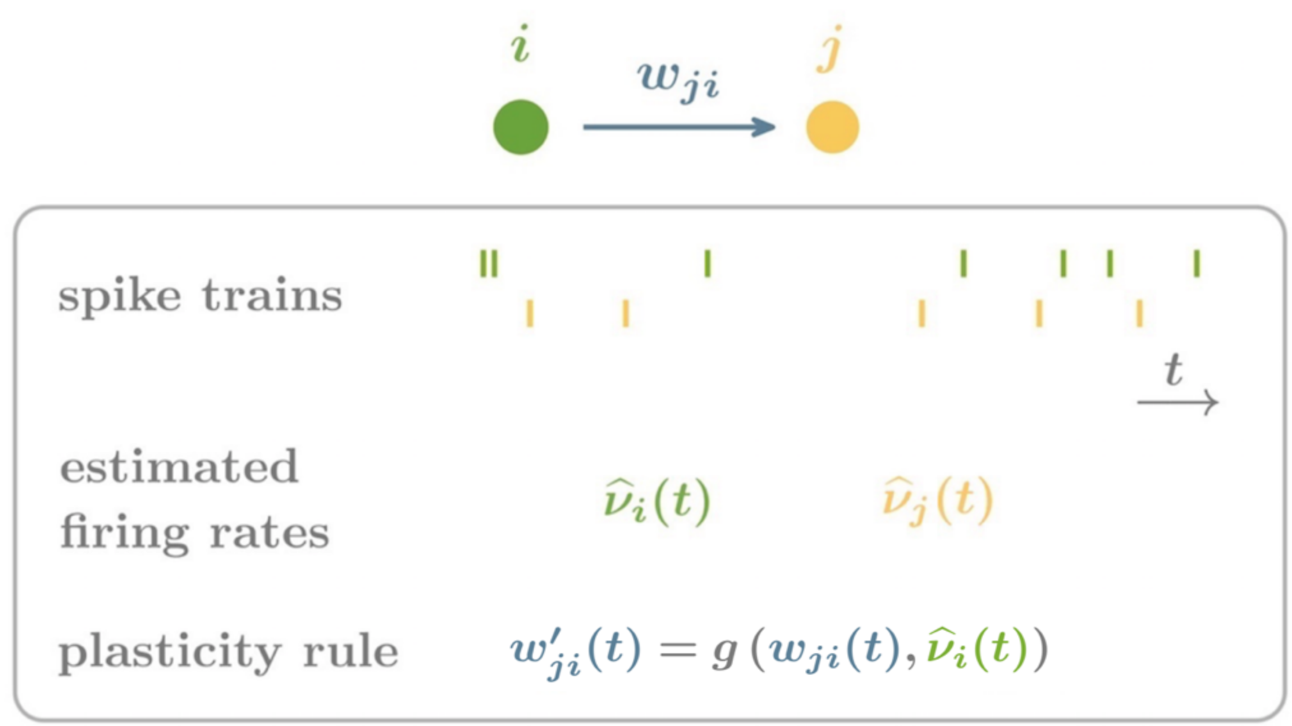


Activity-dependent plasticity rule

Goal:

from

the neuronal dynamics
the connectivity structure
the plasticity rule

infer

the stationary distribution of firing rates

Isolated neuron

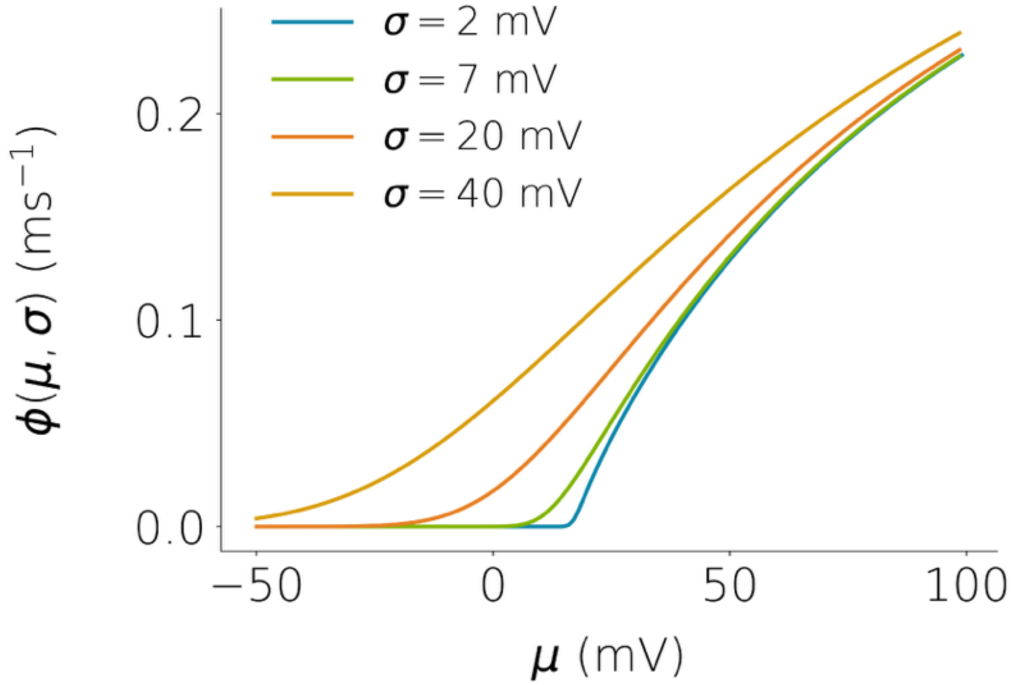
ν	firing rate
K	in-degree
w_i	synaptic weight of i-th input
ν_i	firing rate of i-th input

Isolated neuron

$$\nu = \phi(\mu, \sigma)$$

$$\mu = \sum_{i=1}^{K} w_i \nu_i$$

$$\sigma^2 = \sum_{i=1}^{K} w_i^2 \nu_i$$



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Neuron in a network

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$$\begin{pmatrix} W \\ Z \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}), \qquad \mathbf{\Sigma} = \begin{pmatrix} s_\mu^2 & c_{\mu\sigma} \\ c_{\mu\sigma} & s_\sigma^2 \end{pmatrix}$$

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$$\nu = \phi(\mu, \sigma) = \nu \left(\overbrace{K, W, Z,}^{\text{random variables}} \quad \overbrace{m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}}^{\text{parameters}} \right)$$

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rate distribution ←

parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$

distribution of K and (W, Z)

$\boldsymbol{\nu = \nu(K, W, Z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma})}$

Neuron in a network

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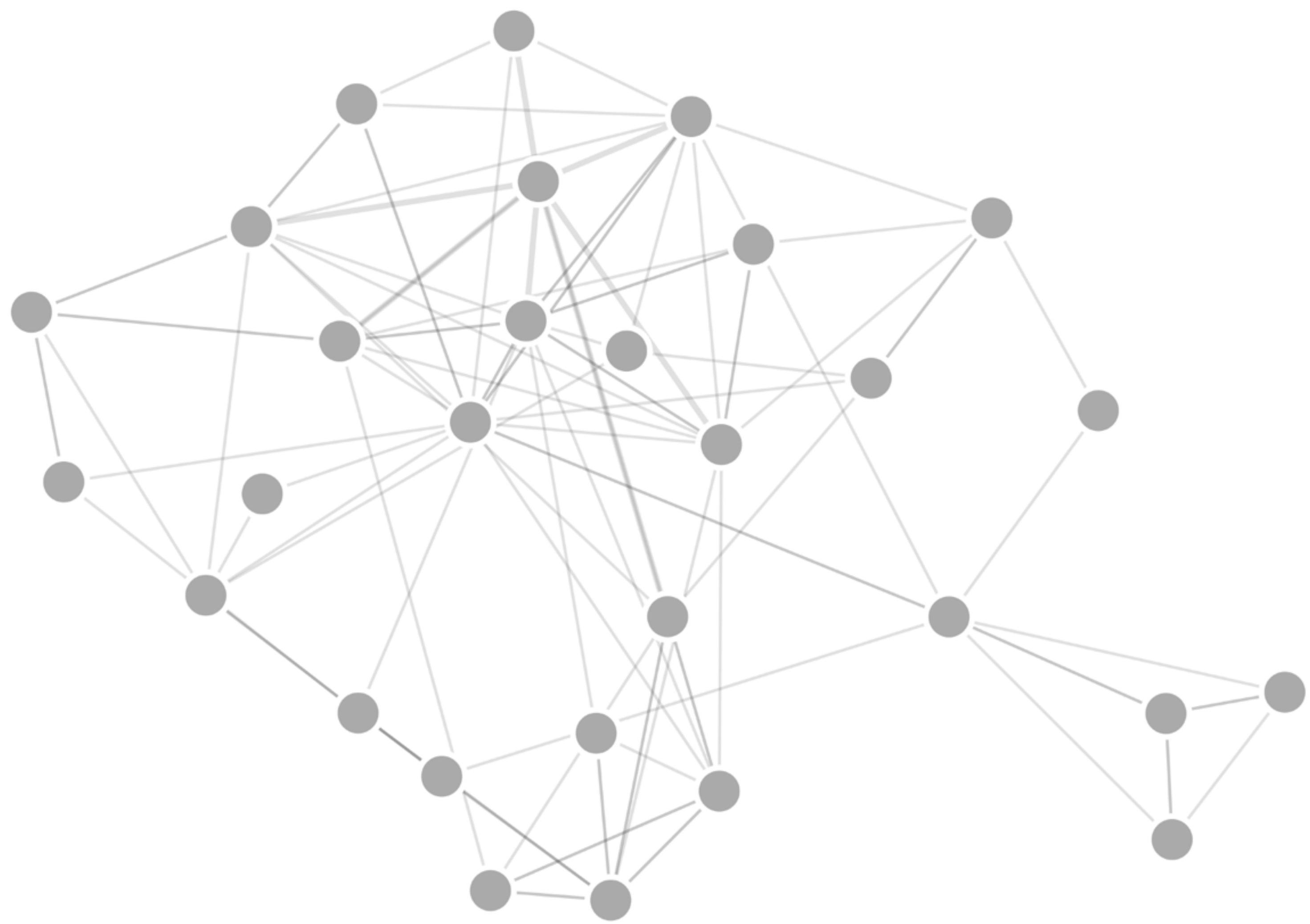
rate distribution ⇄

parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$

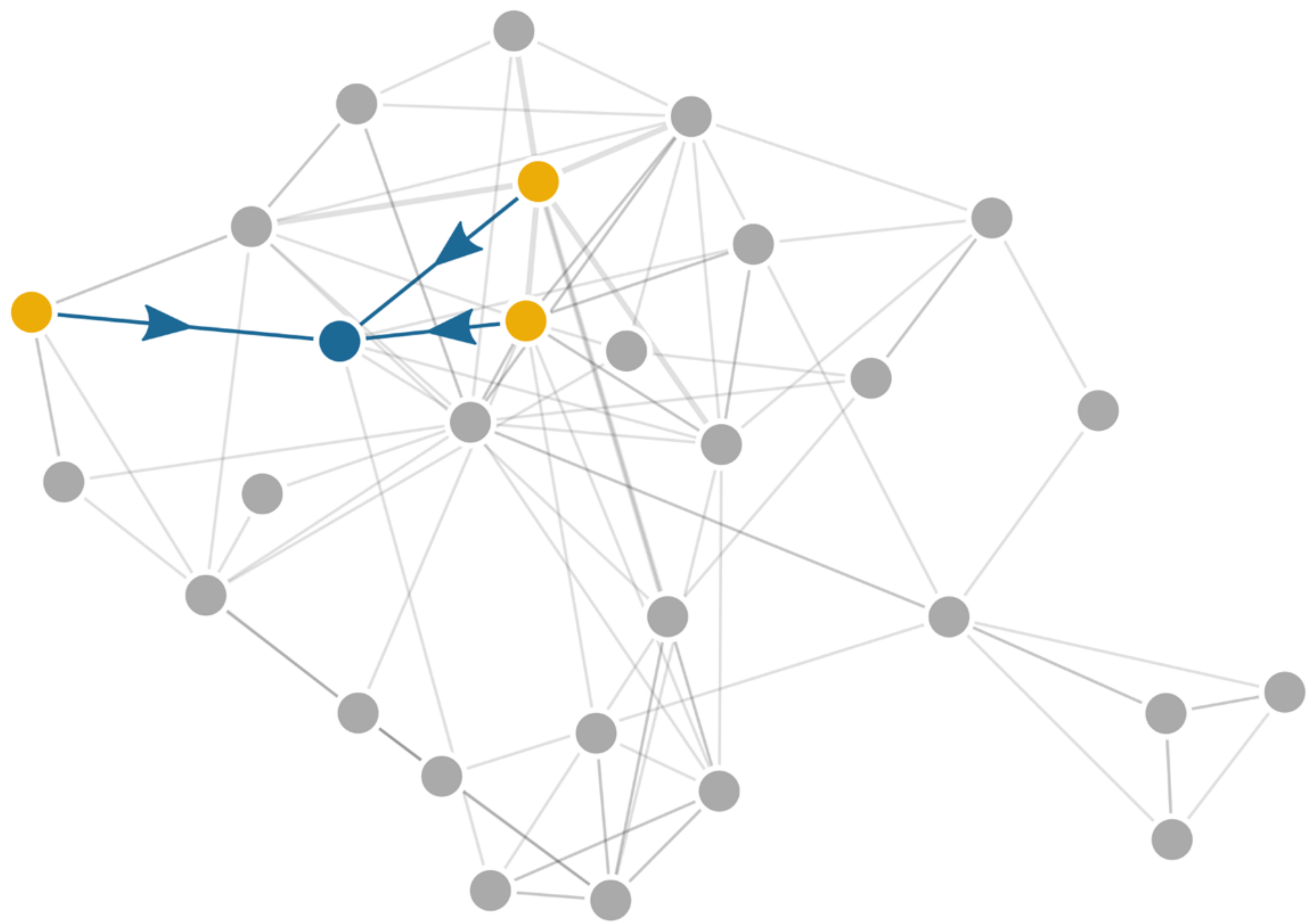
distribution of K and (W, Z)

$$\boldsymbol{\nu = \nu\left(K, W, Z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)}$$

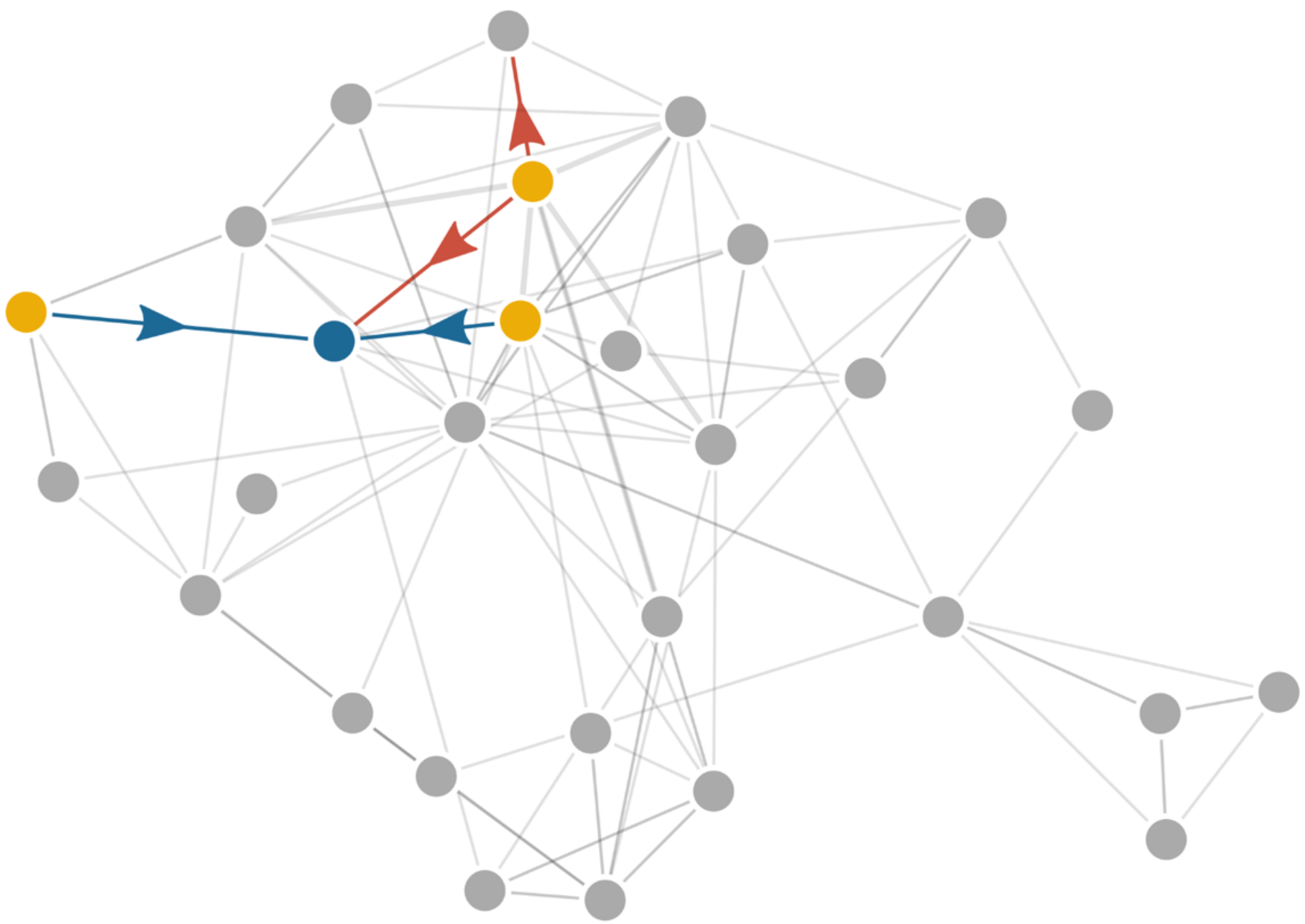
An observation…



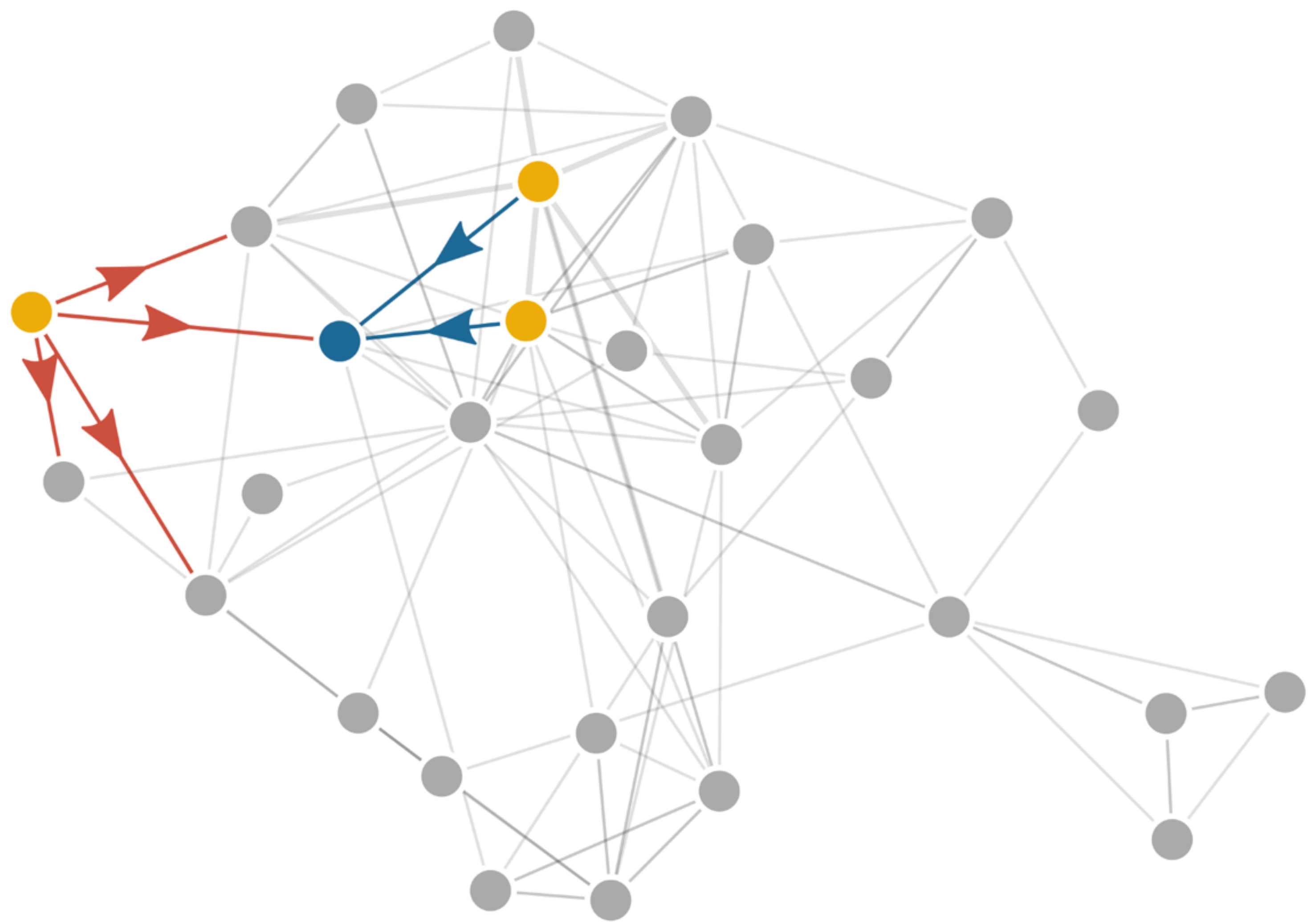
in-neighbors



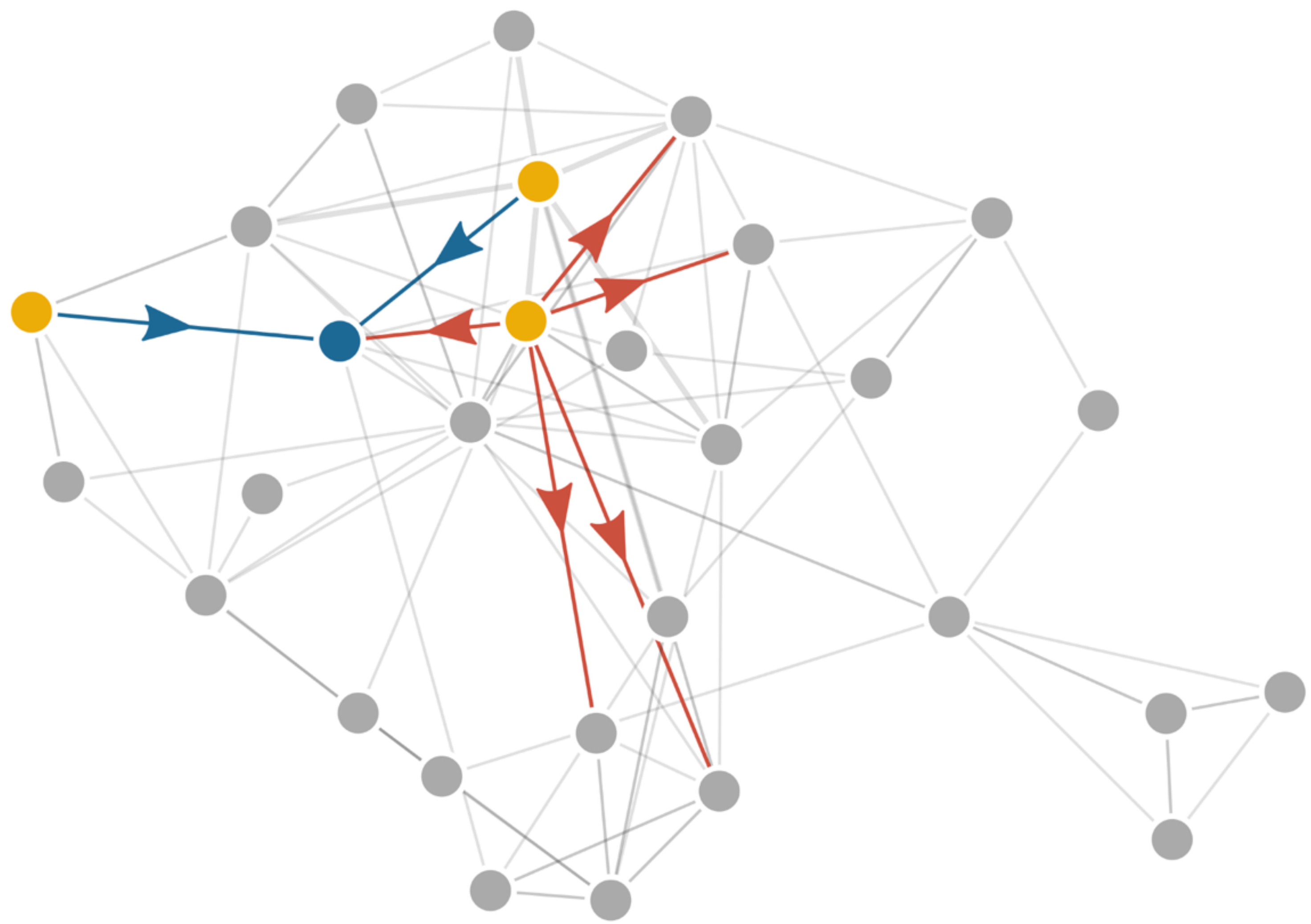
in-neighbors out-degree of in-neighbors



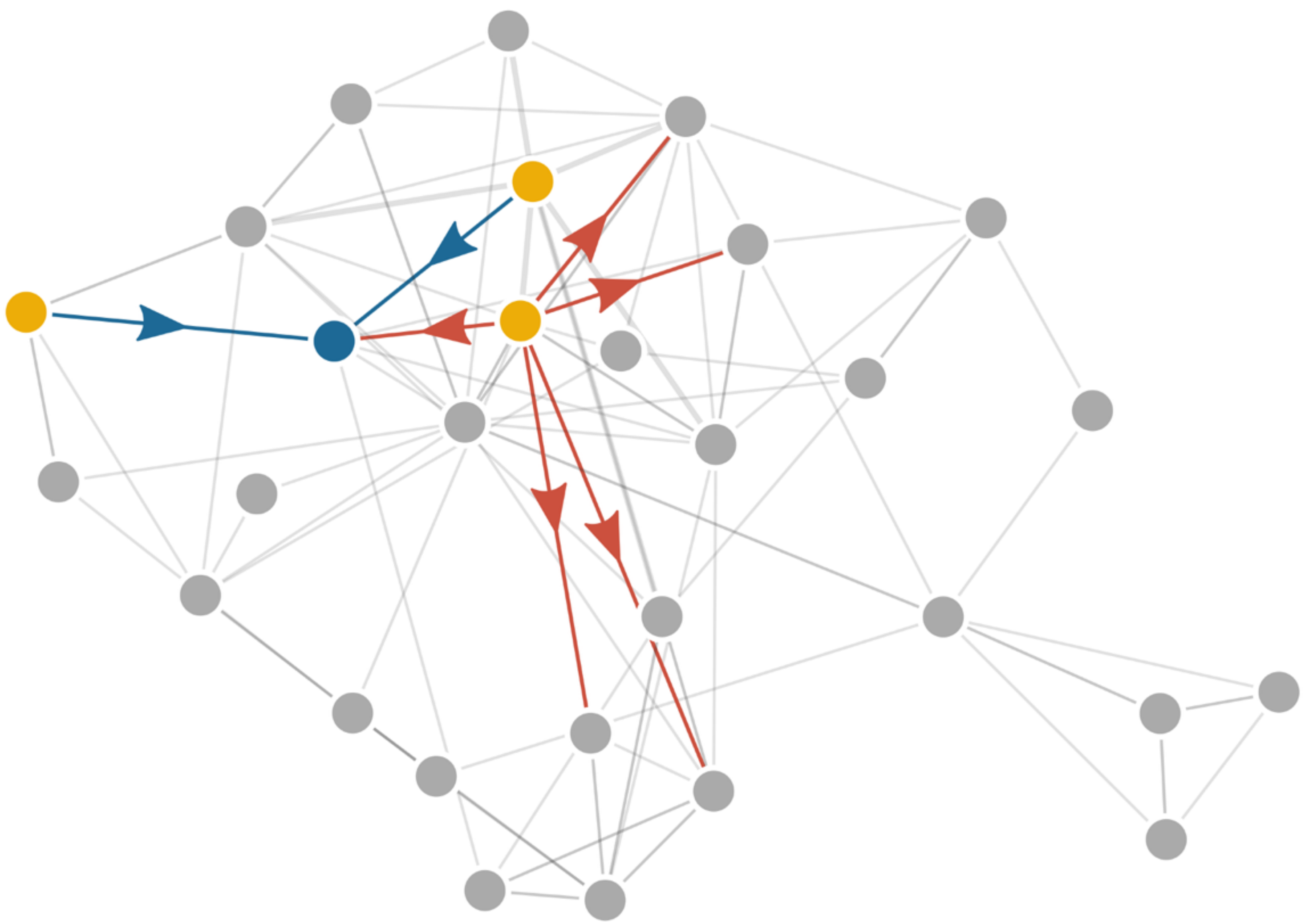
in-neighbors   out-degree of in-neighbors



in-neighbors out-degree of in-neighbors

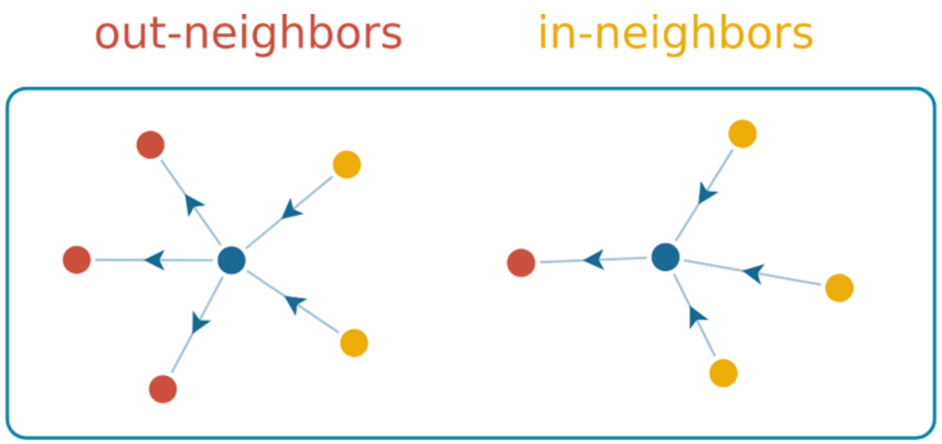


The out-degree of in-neighbors tends to be larger than the out-degree of nodes



r : correlation coefficient between in- and out-degree of nodes

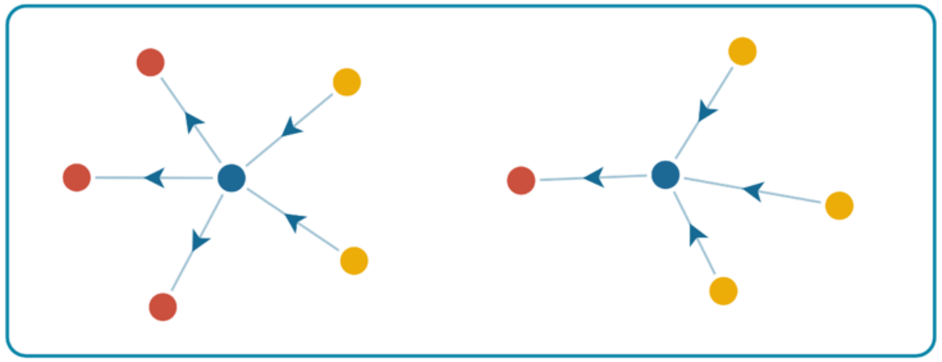
$r = 0$



r : correlation coefficient between in- and out-degree of nodes

out-neighbors in-neighbors

$r = 0$



ρ_K in-degree density

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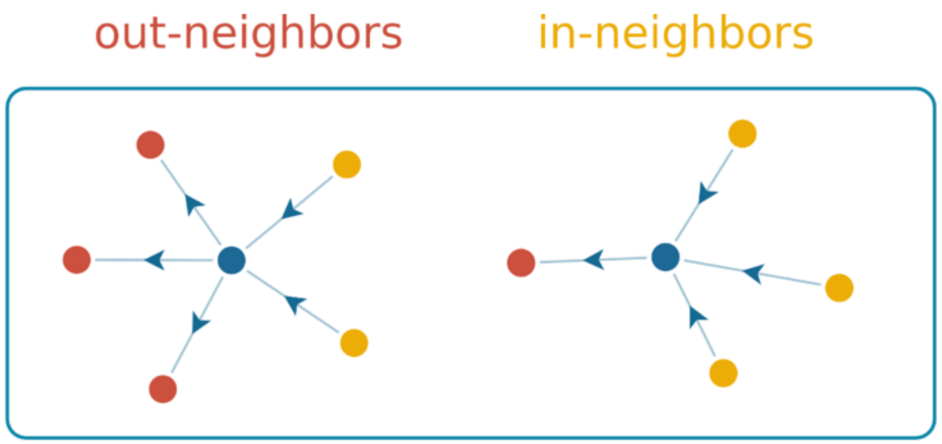
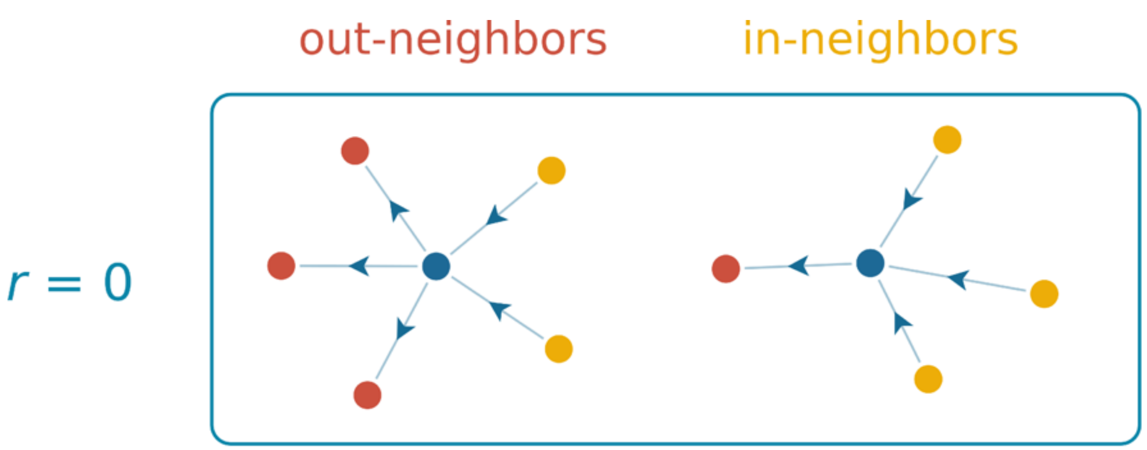
$r = 0$

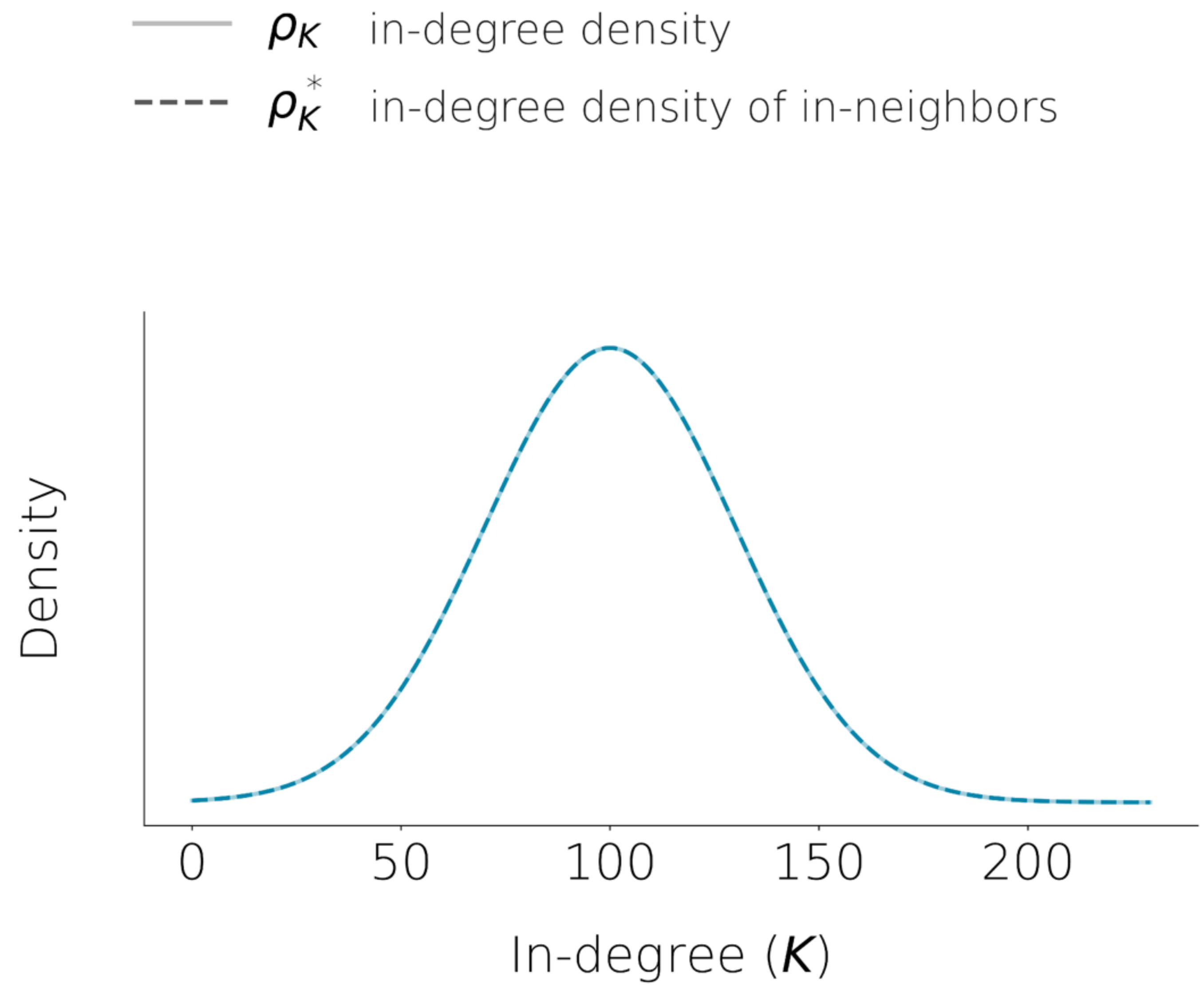


ρ_K in-degree density

ρ_K^* in-degree density of in-neighbors

r : correlation coefficient between in- and out-degree of nodes





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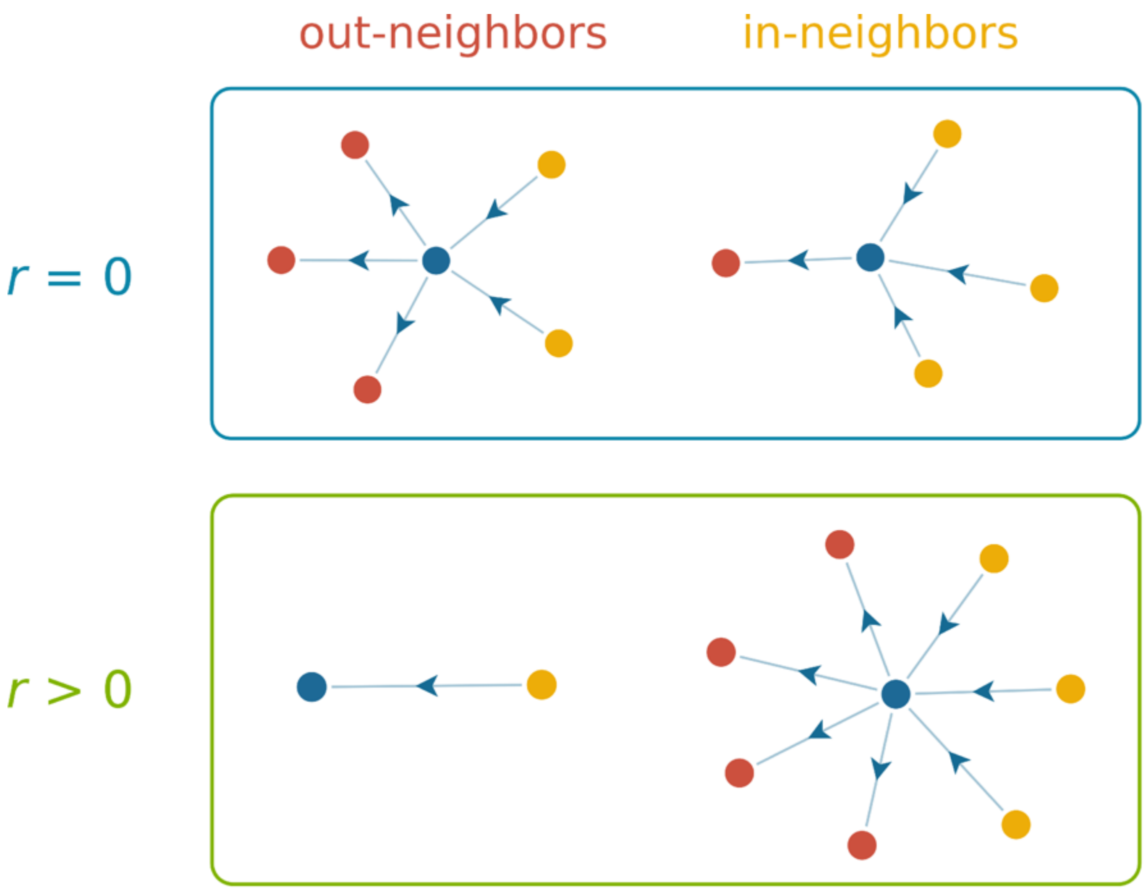

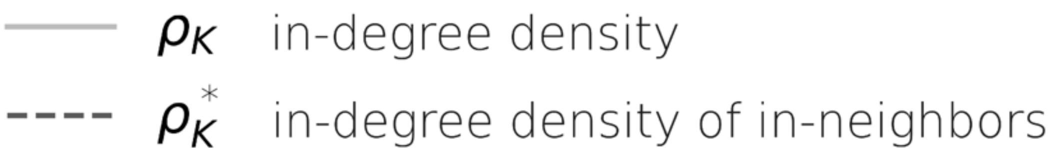

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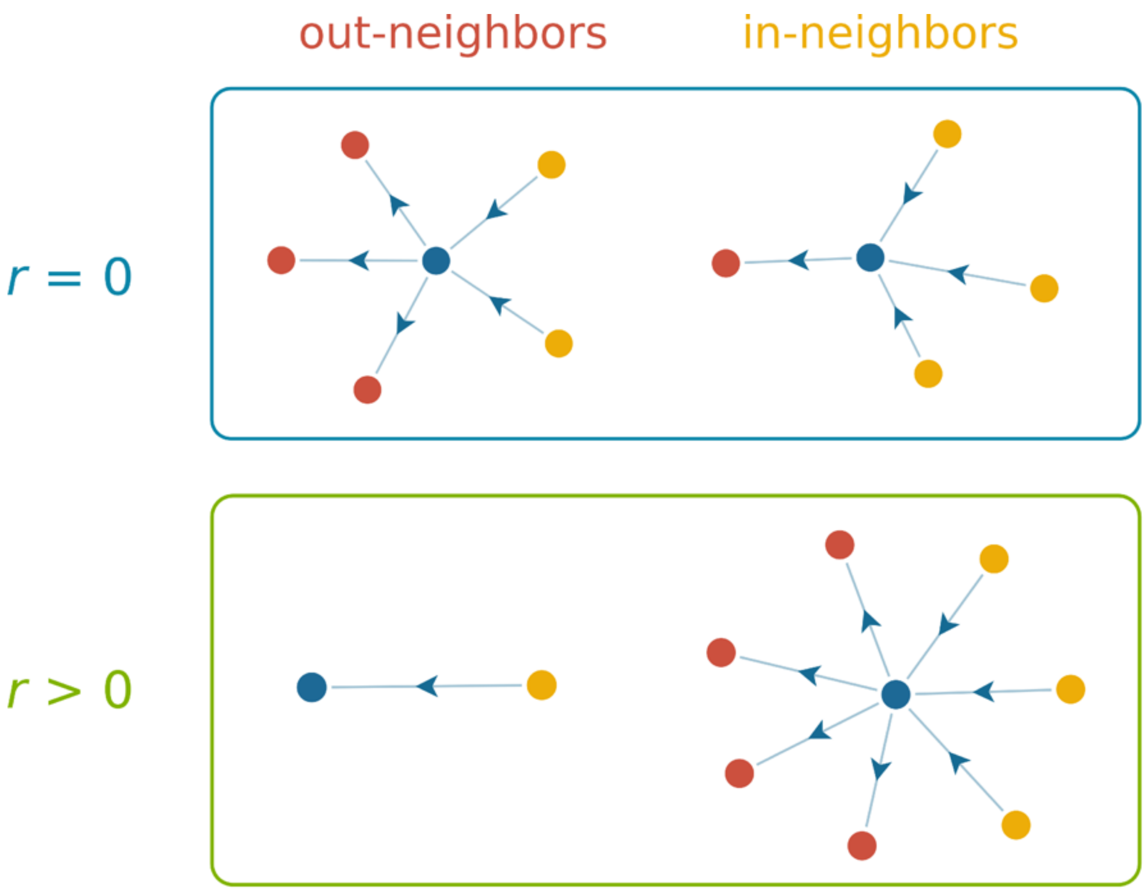

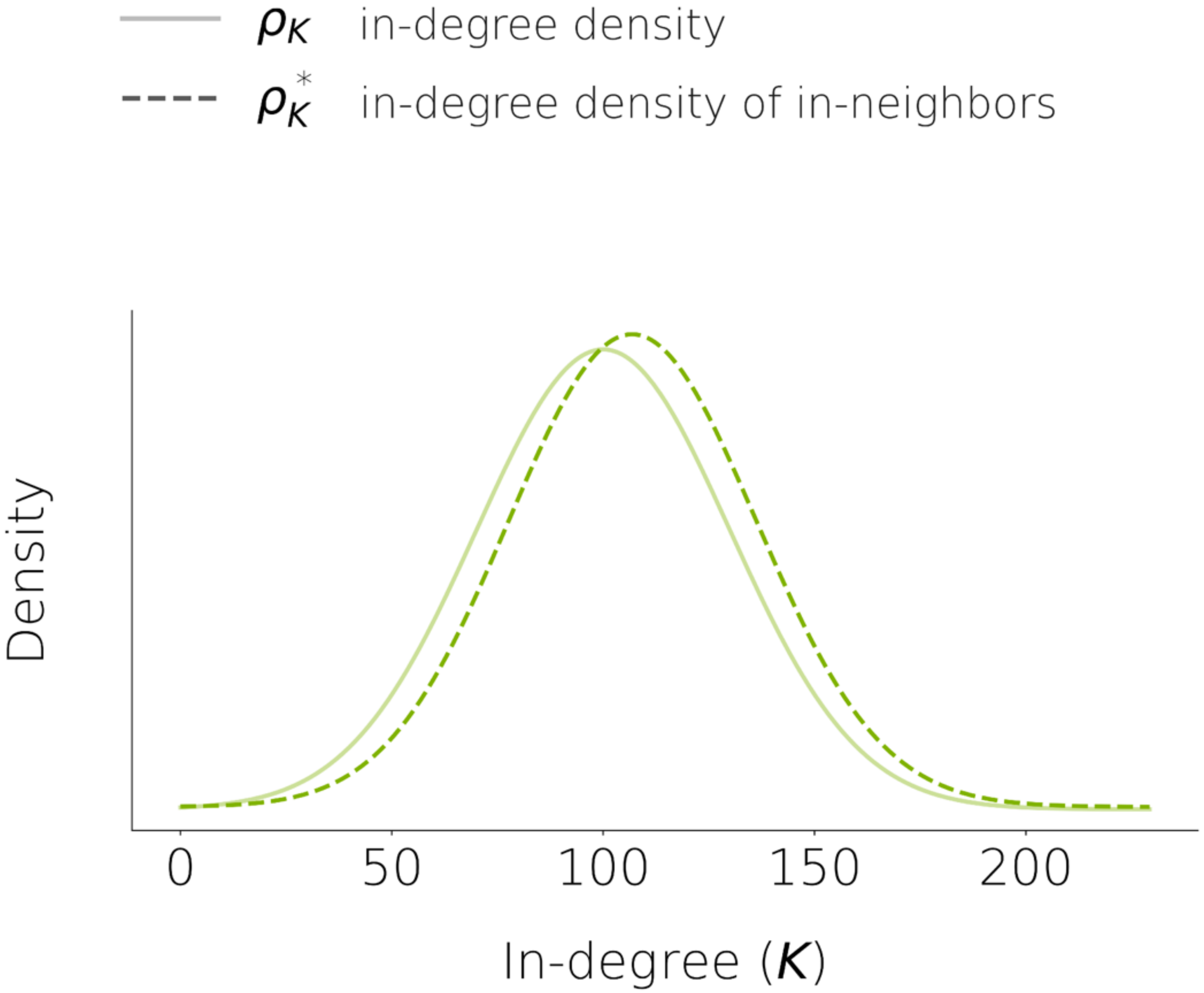

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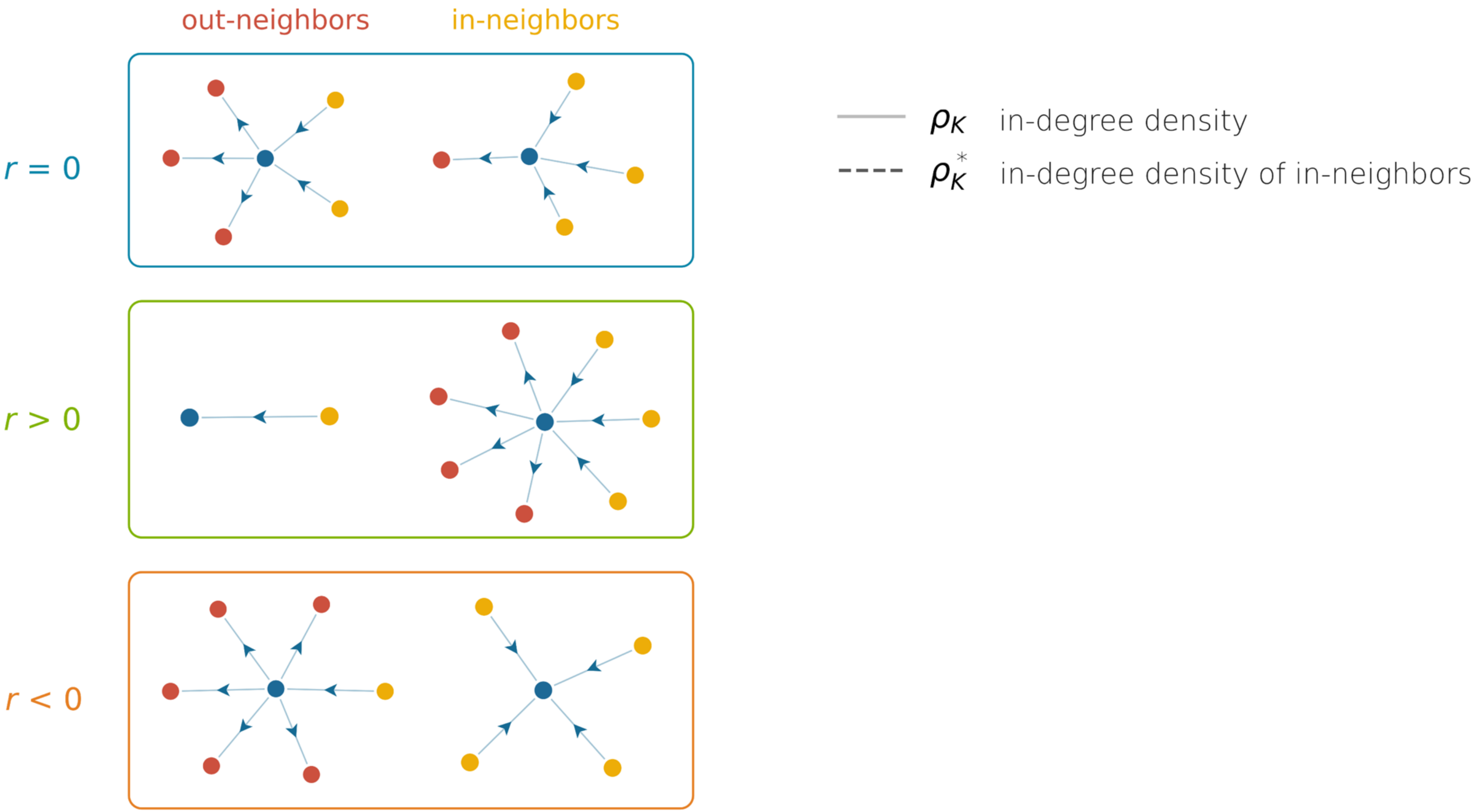

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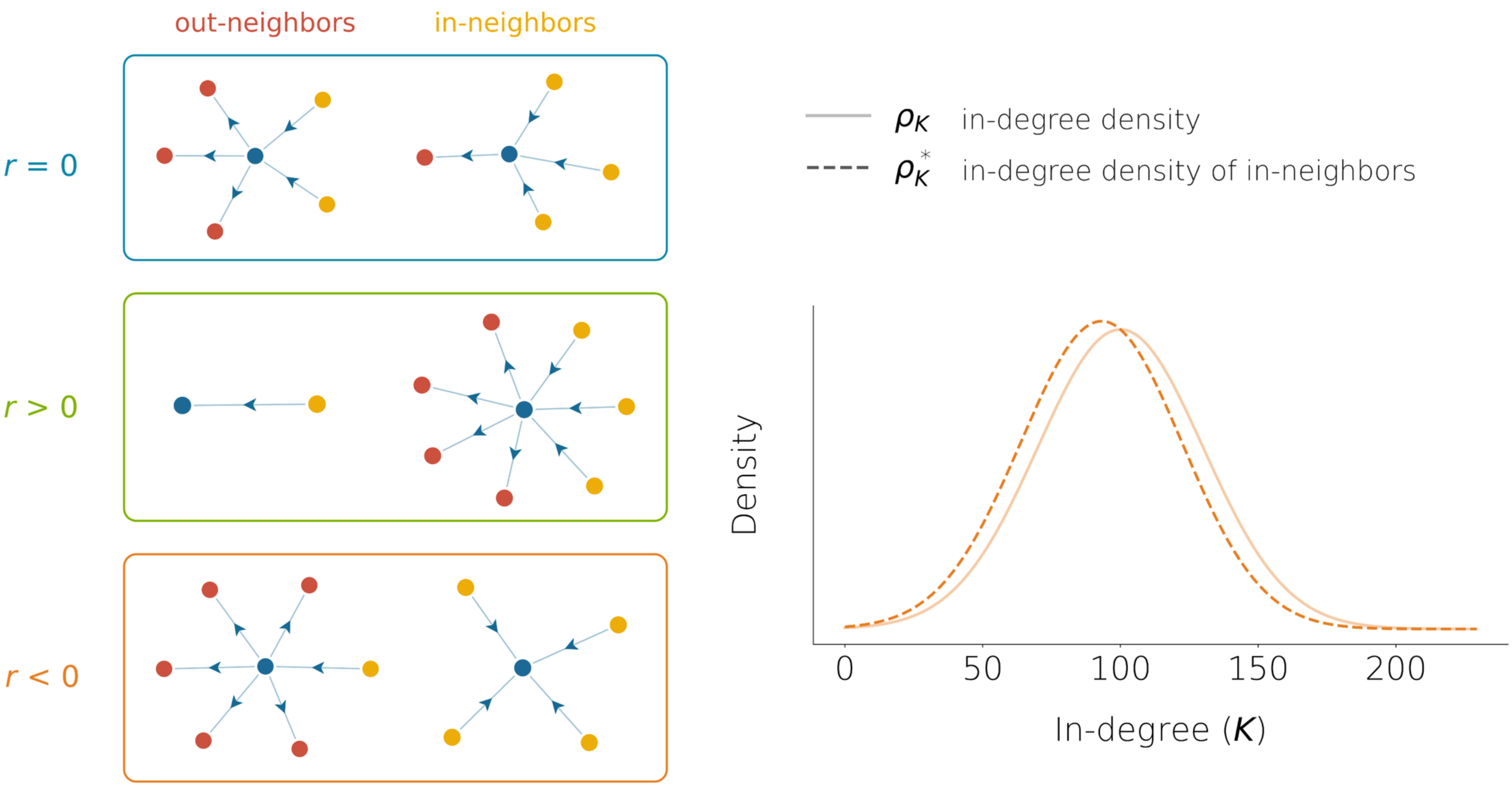

Closing the loop


rate distribution
parameters $m_\mu, m_\sigma, s^2_\mu, s^2_\sigma, c_{\mu\sigma}$

rate distribution ⇄ **parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$**

$$m_\mu = \mathbb{E}[\nu_i w_i]$$

rate distribution ⇄ **parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$**

$$m_\mu = \mathbb{E}[\nu_i w_i]$$

plasticity rule dependent on pre-synaptic activity

$$w_i'(t) = g(w_i(t), \nu_i(t))$$

steady state weight-rate relationship

$$w_i = f(\nu_i)$$

rate distribution ⇄ **parameters** $\boldsymbol{m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}}$

$$\begin{aligned} m_\mu &= \mathbb{E}[\nu_i w_i] \\ &= \mathbb{E}[\nu_i f(\nu_i)] \end{aligned}$$

plasticity rule dependent on pre-synaptic activity

$$w_i'(t) = g(w_i(t), \nu_i(t))$$

steady state weight-rate relationship

$$w_i = f(\nu_i)$$

rate distribution ⇄ **parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$**

$$
\begin{aligned}
m_\mu &= \mathbb{E}[\nu_i w_i] \\
&= \mathbb{E}[\nu_i f(\nu_i)] \\
&= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \nu\left(k, w, z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\, f\left(\nu(k, w, z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma})\right) \\
&\qquad \rho_K^*(k)\, \rho_{W,Z}(w, z)\, \mathrm{d}w\, \mathrm{d}z\, \mathrm{d}k
\end{aligned}
$$

$$
\begin{aligned}
m_\mu &= \mathbb{E}[\nu_i w_i] \\
&= \mathbb{E}[\nu_i f(\nu_i)] \\
&= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \nu\left(k, w, z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\; f\left(\nu(k, w, z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma})\right) \\
&\qquad \rho_K^*(k)\, \rho_{W,Z}(w, z)\, \mathrm{d}w\, \mathrm{d}z\, \mathrm{d}k \\
&= F_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)
\end{aligned}
$$

$$m_\mu = F_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)$$

rate distribution ⇄ **parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$**

$$
\begin{aligned}
m_\mu &= F_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\\
m_\sigma &= F_\sigma\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\\
s_\mu^2 &= G_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\\
s_\sigma^2 &= G_\sigma\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)\\
c_{\mu\sigma} &= H\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)
\end{aligned}
$$

$$
\begin{cases}
m_\mu = F_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right) \\
m_\sigma = F_\sigma\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right) \\
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\end{cases}
$$

rate distribution ⇄ **parameters $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$**

$$
(*)\ \begin{cases}
m_\mu = F_\mu\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right) \\
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c_{\mu\sigma} = H\left(m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)
\end{cases}
$$

Solve (*) for the unknowns $m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}$

Once the parameters $m_\mu, m_\sigma, s^2_\mu, s^2_\sigma, c_{\mu\sigma}$ are computed:

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the distribution of K and (W, Z) is known

and

the firing rate of a neuron with $K = k, (W = w, Z = z)$ can be computed through

$$\nu = \nu\left(k, w, z, m_\mu, m_\sigma, s_\mu^2, s_\sigma^2, c_{\mu\sigma}\right)$$

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This allows us to reconstruct the firing rate distribution

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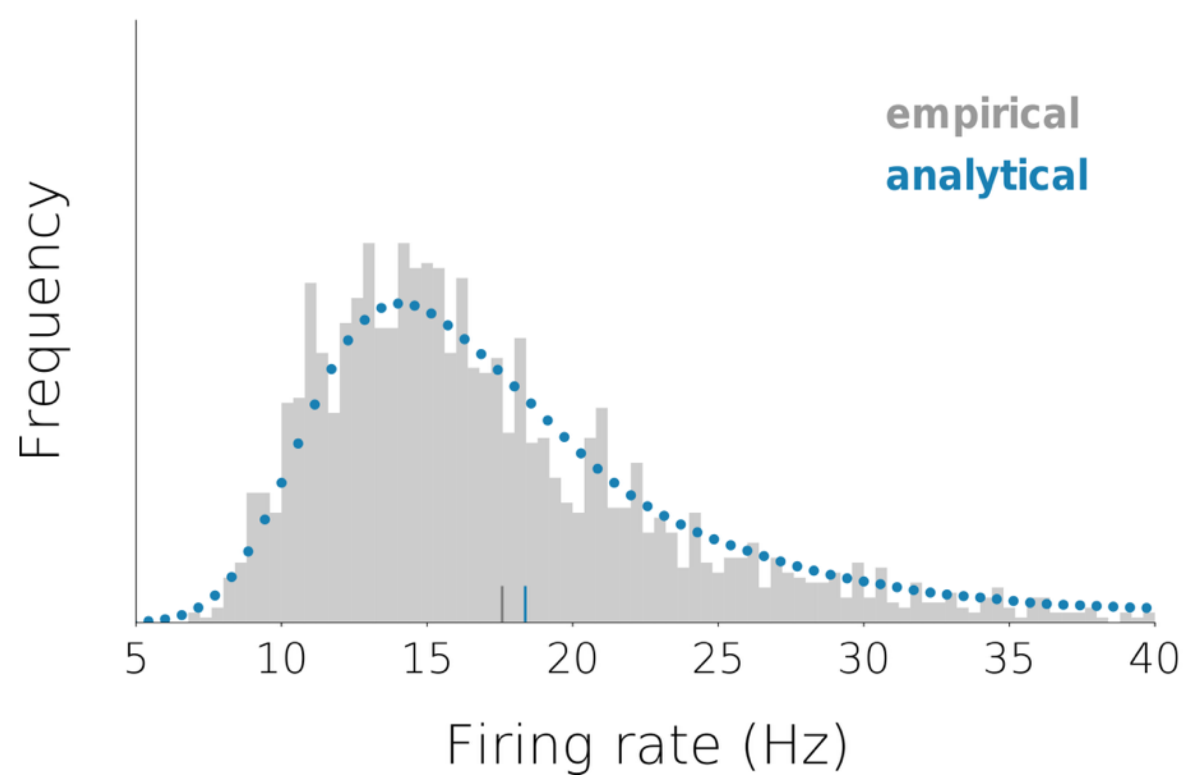

This formalism …

can be extended to networks

with different neuronal populations

with plasticity rules dependent on pre- and post-synaptic activities

and can help to

explore the way in which plasticity shapes activity in neuronal networks