Threefold way to the dimension reduction of dynamics on networks: an application to synchronization

V. Thibeault, G. St-Onge, L. J. Dubé, and P. Desrosiers

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Picturing a complex system as a whole and forecasting its long-term evolution often looks like an impossible task. Yet, behind the high-dimensional nonlinear dynamics and the intricate organization that characterize complex systems, there are essential mechanisms that explain the emergence of macroscopic phenomena.

We propose a Dynamics Approximate Reduction Technique (DART) that maps high-dimensional (complete) dynamics unto low-dimensional (reduced) dynamics while preserving the most salient topological and dynamical features of the original system. DART generalizes previous approaches [2] and is used to predict the emergence of synchronization [1].

DART: Dynamics Approximate Reduction Technique

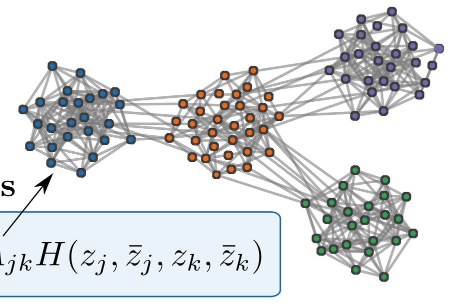
Definitions

	Complete dynamics	Reduced dynamics
Dimension of the dynamics and number of nodes	$N \gg 1$	$n < N$
Indices	Latin $j \in \{1,...,N\}$	Greek $\mu \in \{1,...,n\}$
Dynamical variable	z_j	Z_μ
Adjacency matrix	A	$\mathcal{A}$
Degree	k_j	κ_μ
Dynamical parameter	ω_j	Ω_μ
Dynamical parameter matrix	W	$\mathcal{W}$
Degree matrix	K	$\mathcal{K}$
Function describing the intrinsic dynamics	F, G	
Function describing the coupling between nodes	H	
Reduction matrix ($n \times N$)	M	

Method

Complex conjugation

Time derivative



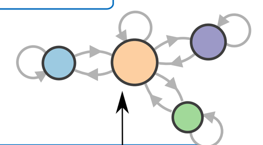
Complete dynamics

$$\dot{z}_j = F(z_j,\bar{z}_j) + \omega_j G(z_j,\bar{z}_j) + \sum_{k=1}^N A_{jk} H(z_j,\bar{z}_j,z_k,\bar{z}_k)$$

Step 1: **Define observables**
Weighted means: $Z_\mu = \sum_{j=1}^N M_{\mu j} z_j$

Step 2: **Differentiate with respect to time**
$$\dot{Z}_\mu = \sum_{j=1}^N M_{\mu j} F(z_j,\bar{z}_j) + \sum_{j=1}^N M_{\mu j}\,\omega_j\, G(z_j,\bar{z}_j) + \sum_{j,k=1}^N M_{\mu j} A_{jk} H(z_j,\bar{z}_j,z_k,\bar{z}_k)$$

Step 3: **Close the system** in terms of the observables



Reduced dynamics

$$\dot{Z}_\mu \approx F(Z_\mu,\bar{Z}_\mu) + \Omega_\mu G(\gamma_\mu,\bar{\gamma}_\mu) + \kappa_\mu H(\delta_\mu,\bar{\delta}_\mu,\epsilon_\mu,\bar{\epsilon}_\mu)$$

with

$$\gamma_\mu = \Omega_\mu^{-1}\sum_{\nu=1}^n \mathcal{W}_{\mu\nu} Z_\nu \qquad \delta_\mu = \kappa_\mu^{-1}\sum_{\nu=1}^n \mathcal{K}_{\mu\nu} Z_\nu \qquad \epsilon_\mu = \kappa_\mu^{-1}\sum_{\nu=1}^n \mathcal{A}_{\mu\nu} Z_\nu$$

Concrete example: DART

Kuramoto model on networks

σ : Coupling constant between the oscillators

ω_j : Natural frequency of oscillator j

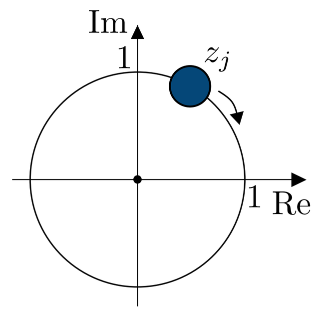


FIG. 1. In a **phase dynamics**, each oscillator j at position z_j is rotating around the unit circle in the complex plane.

$$\dot{z}_j = i\omega_j z_j + \frac{\sigma}{N}\sum_{k=1}^N A_{jk}[z_j - z_j^2 \bar{z}_k]$$

DART

$$\dot{Z}_\mu = i\sum_{\nu=1}^n \mathcal{W}_{\mu\nu} Z_\nu + \frac{\sigma}{2N}\sum_{\nu=1}^n \mathcal{A}_{\mu\nu} Z_\nu - \frac{\sigma}{2N\kappa_\mu^2}\sum_{\nu,\xi,\tau=1}^n \mathcal{A}_{\mu\nu}\mathcal{K}_{\mu\xi}\mathcal{K}_{\mu\tau} Z_\xi Z_\tau \bar{Z}_\nu$$

DART can also be applied to other phase dynamics, such as the Winfree and theta models [1], and to other nonlinear dynamics on networks [2].

Construction of the reduction matrix

Threefold problem

To close the system, we need to solve three **compatibility equations**:

$\mathcal{W}M = MW$ (Dynamical parameters)

$\mathcal{K}M = MK$ (Local structure)

$\mathcal{A}M = MA$ (Global structure)

How to choose the reduction matrix M ?
Combine eigenvectors of W, K, or A.

$M = CV$

C: $n \times n$ coefficient matrix

V: $n \times N$ eigenvector matrix

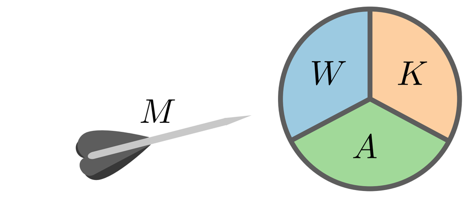


FIG. 2. In DART, we need to aim one target matrix W, K, or A to solve its compatibility equation.

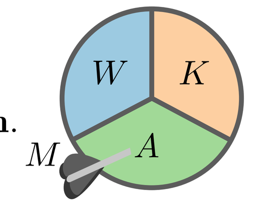
Once M is chosen, the best solution to the compatibility equations is

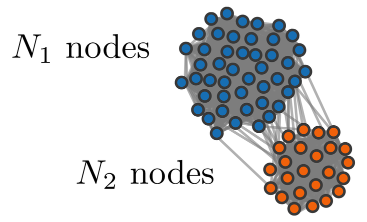
$$\mathcal{W} = MWM^+ \qquad \mathcal{K} = MKM^+ \qquad \mathcal{A} = MAM^+$$

where $^+$ is the Moore-Penrose pseudo-inversion.

Concrete example: construct M

Let us target A drawn from a **random modular graph**.



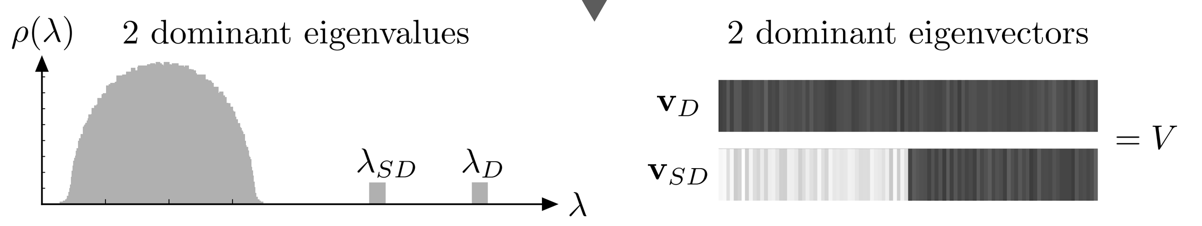


Stochastic block model (SBM)

$p_{\mu\nu}$: Probability that a node in community μ is connected to a node in community ν

If $p_{\mu\mu} = 0$ for all μ, it is a **random bipartite graph**.

Get V



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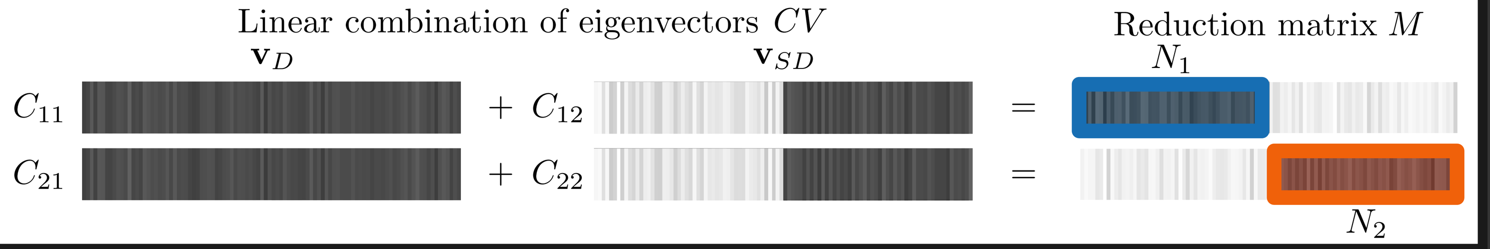

Application to synchronization

Phase synchronization observable

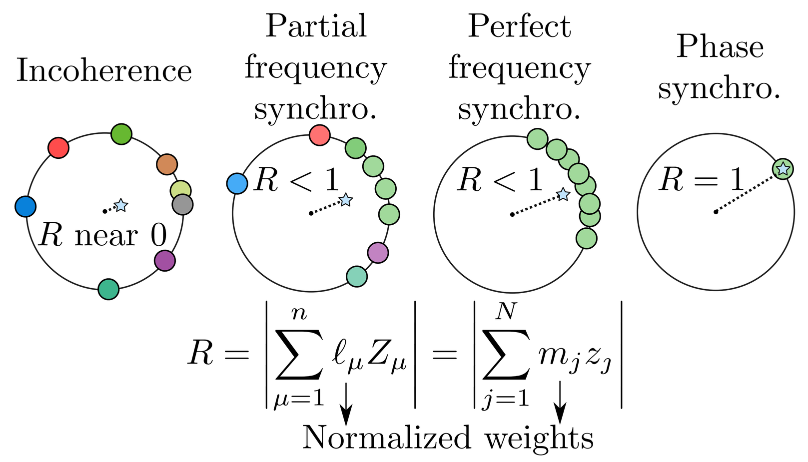

FIG. 3. Global synchronization observable for phase dynamics. Different node colors represent different natural frequencies.

$\langle\cdot\rangle_t$: Average over time

$\langle\cdot\rangle$: Average over time, graphs, dynamical parameters, initial conditions

Predict synchronization on random modular graphs

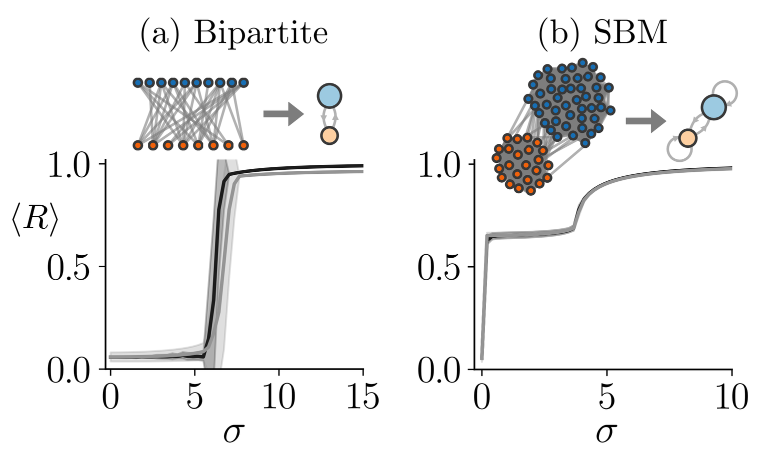

FIG. 4. Synchronization curve of the complete ($N = 250$, black lines) vs. reduced ($n = 2$, gray lines) Kuramoto dynamics on random modular graphs.

Predict bifurcations to chimeras

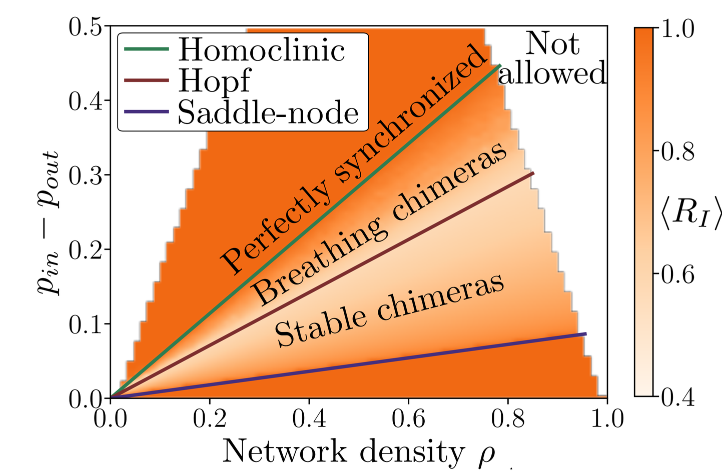

FIG. 5. Chimera state regions in the Kuramoto-Sakaguchi dynamics on the mean SBM. Each point represents the time average of the phase synchronization observable of the incoherent community obtained with the integration of the complete dynamics ($N = 500$). The Hopf and saddle-node bifurcations are obtained from the reduced dynamics ($n = 2$). $p_{11} = p_{22} = p_{in}$, $p_{12} = p_{21} = p_{out}$

Existence of periphery chimeras

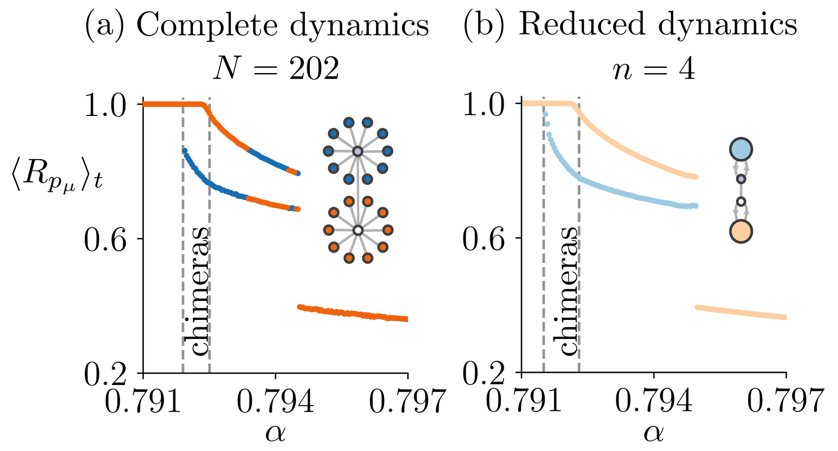

FIG. 6. Periphery chimeras exist for the Kuramoto-Sakaguchi dynamics on two star graphs. The existence of these chimeras is restricted to a small range of phase lags α (between the two vertical dashed lines). The time-averaged synchronization observable in periphery p_μ is denoted $\langle R_{p_\mu}\rangle$.

Predict explosive synchronization

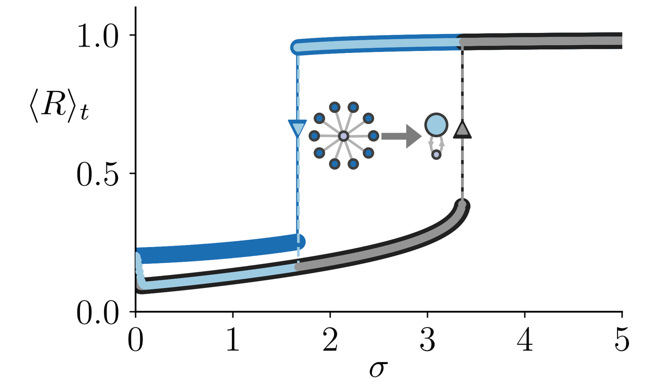

FIG. 7. Hysteresis in the Kuramoto-Sakaguchi dynamics on the star graph. Complete dynamics: $N = 11$, dark blue (backward branch) and black (forward branch) markers. Reduced dynamics: $n = 2$, light blue (backward branch) and gray (forward branch) markers.

Future works

Challenges

- Apply DART to dynamics on weighted, directed, and real networks.
- Generalize DART for nonlinear observables.
- Relate DART to existing dimension-reduction methods.

Coming soon

- Find better algorithms to solve the compatibility equations.
- Apply DART to plant-pollinisator dynamics on bipartite networks.
- Apply DART to nonlinear neural dynamics with adaptation.

For more details, see the paper [1] !

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[1] V. Thibeault, G. St-Onge, L. J. Dubé, and P. Desrosiers, *arXiv:2005.10922*, (2020).

[2] E. Laurence, N. Doyon, L. J. Dubé, and P. Desrosiers, *Phys. Rev. X*, **9**, p.011042 (2019).